Martin Ráhus B 612 19.9.

# Cvičenie 1 –

## Delenie signálov a ich charakteristiky

signály: energetické – nekoná prácu (energia), ale umožňujú detekovať signál
výkonové – najviac na strane vysielača nás zaujímajú

1, výkonové

$P_x$ – stredný normovaný výkon [W]

$P_x > 0$

$P_x < \infty$

$$P = U \cdot I = I^2 \cdot R = \frac{U^2}{R}$$

ak $R = 1\,\Omega \rightarrow P_x = x^2$

ak $\bar{x}(t)$ – komplexný signál:

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |\bar{x}(t)|^2 \, dt$$

pre reálne signály:

$$P_x = \lim \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t) \, dt$$

patrí sem:
- periodické signály
- kvázipeviodické sig.
- stochastické sig.

2, energetické $0 < E_x < \infty$

$E_x$ – stredná normovaná energia [J]

pre reálne sig.:

$$E_x = \int_{-\infty}^{\infty} x^2(t) \, dt$$

patrí sem:
- deterministické signály
- neperiodické sig.

3) <u>výkonová spektrálna hustota</u>

Parsevalova rovnosť (reálne sig.)

$$P_x = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df$$

$$x(t) \overset{\hat{F}R}{\longleftrightarrow} X(f)$$

$$PSD = |X(f)|^2 = G_x(f) \Rightarrow G_x(f) \text{ - nezáporná a párna}$$

$G_x$ - výkonová spektrálna hustota $\left[\frac{W}{Hz}\right]$

$$G_x(f) = \sum_{n=-\infty} |c_n|^2\,\delta(f - nf_0)$$

$c_n$ = koef. FR

$$P_x = \underbrace{\int_{-\infty}^{\infty} G_x(f)\,df}_{\text{jednostr.}} = \underbrace{2\int_{0}^{\infty} G_x(f)\,df}_{\text{dvojstr.}}$$

PSD: power spectral density = výkonová hustota

4) Energetická spektrálna hustota - ESD

$$E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df$$

$$ESD = |X(f)|^2 = \Psi_x(f)\left[\frac{J}{Hz}\right]$$

$$E_x = \underbrace{\int_{-\infty}^{\infty} \Psi_x(f)\,df}_{\text{dvojstr.}} = \underbrace{2\int_{0}^{\infty} \Psi_x(f)\,df}_{\text{jednostr.}}$$

<u>Pr:</u> Vypoč. stred. norm. výkon

$$x(t) = A\cos 2\pi f t$$

a) v časovej oblasti $\qquad P_x = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)\,dt$

b) vo frekvenčnej -||-

$$P_x = \frac{1}{T} \cdot A \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos^2 2\pi f t \, dt$$

$$\cos^2 x = \frac{1}{2} \cdot (1 + \cos 2x)$$

$$P_x = \frac{A^2}{2T} \cdot \left( \int_{-\frac{T}{2}}^{\frac{T}{2}} 1 \, dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos 4\pi f t \, dt \right) = \frac{A^2}{2T} \left\{ \left( \frac{T}{2} + \frac{T}{2} \right) + \left[ \frac{\sin 4\pi f t}{4\pi f} \right]_{-\frac{T}{2}}^{\frac{T}{2}} \right.$$

$$= \frac{A^2}{2} + \frac{\sin 2\pi f T + \sin 2\pi f T}{4\pi f} = \frac{A^2}{2} + \frac{\cancel{2} \sin 2\pi f T}{\cancel{4}\pi f} = \frac{A^2}{2} + \frac{\sin 2\pi f T}{2\pi f}$$

$$\boxed{P_x = \frac{A^2}{2}}$$

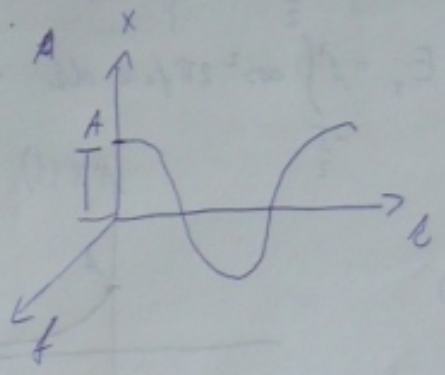
$$c_n = \begin{cases} \frac{1}{2}(a_n - j b_n) & n > 0 \\ \frac{a_0}{2} & n = 0 \\ \frac{1}{2}(a_n + j b_n) & n < 0 \end{cases}$$

$$|c_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$a_0 = 0$

$b_n = 0$

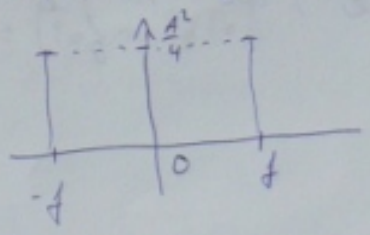
$a_1 = A$




$P_x =$

$$P_x = |c_1|^2 + |c_{-1}|^2 = 2 \cdot \left| \frac{A}{2} \right|^2 = \frac{A^2}{2}$$

$$c_1 = c_{-1} = \frac{A}{2}$$



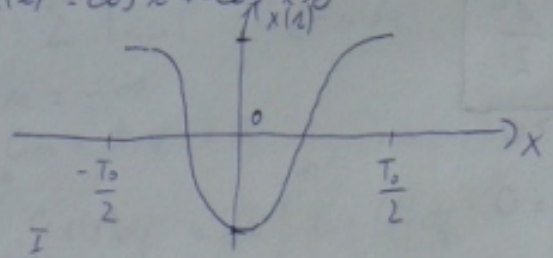
Pr. 2: Určte, kt. z nasledujúcich signálov je výkonový alebo energetický a vypočítajte príslušné veličiny:

a) $x(t) = \begin{cases} A\cos 2\pi f_0 t & -\frac{T_0}{2} \le t \le \frac{T_0}{2} \\ 0 & \text{inak} \end{cases}$

b) $x(t) = \begin{cases} A\exp(-at) & \\ 0 & \text{inak} \end{cases}$ a: konst. $a>0$ $t>0$

c) $x(t) = \cos t + 5\cos 2t$

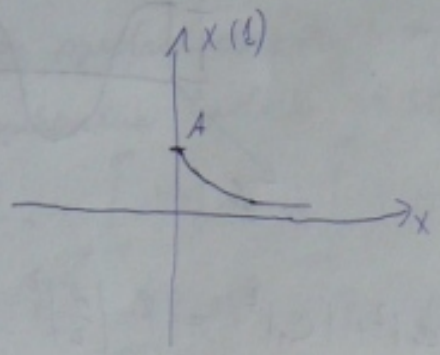
a)




- energetický

$$E_x = A^2\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos^2 2\pi f_0 t \, dt = \frac{A^2 T_0}{2}$$

b)




- energetický

$$E_x = \int_0^\infty A^2 (e^{-at})^2 \, dt = A^2 \int_0^\infty e^{-at2} \, dt = A^2 \left[ \frac{e^{-2at}}{-2a} \right]_0^\infty = \frac{A^2}{2a}$$

c) - výkonový

$|c_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$

$c_1 = \frac{1}{2} \qquad c_2 = \frac{5}{2} \qquad \cancel{|c_n| = \frac{1}{2}}$

$c_{-1} = \frac{1}{2} \qquad c_{-2} = \frac{5}{2}$

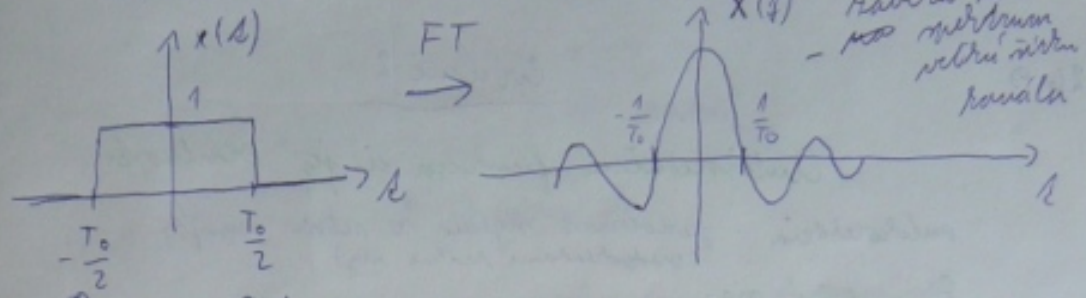
$$P_x = 2|c_1|^2 + 2|c_2|^2 = 13$$

<u>Př 3:</u> sig. $x(t) = \text{rect}\left|\frac{t}{T_0}\right|$ - energ. sig.

vyp. a) ESD

b) $E_x$

$ESD = |X(f)|^2$




$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

cosxa

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$X(f) = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\overbrace{2\pi f}^{\omega} t}\, dt = \frac{e^{-j\pi f T} - e^{j\pi f T}}{-j2\pi f} = \cancel{\text{sinc}}\ T_0 \frac{\sin \omega \frac{T_0}{2}}{\omega \frac{T_0}{2}} = T_0\, si\, \pi f T_0$$

$$ESD = (T_0\, si\, \pi f T_0)^2$$

b) $E_x = T_0$

$$\text{sinc}\, x = \frac{\sin(\pi x)}{\pi x}$$

$$si\, x = \frac{\sin x}{x}$$

Př.4: $P_x = ?$

$$x(t) = 10\cos 10t + i\,10\sin 10t + 20\cos 20t$$

$$c_n = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$$c_1 = \frac{\sqrt{2}}{2}\,10 = \frac{10}{\sqrt{2}}$$

$$c_2 = \frac{10}{\sqrt{2}}$$

$$P_x = \cancel{\frac{100}{2} + \frac{100}{2} = 150}\; 2\cdot\left(\frac{100}{2} + 100\right) = (150)\cdot 2 = 300$$

26.9.

## Cvičení 2

### Autokorelační funkci a jej vlastnosti

autokorelácia - podobnosť signálu so sebou samým
(nedeterministické reálne sig.)

Energetické sig.

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\,x(t+\tau)\,dt$$

Vlastnosti:

1, $R_x(\tau) = R_x(-\tau)$ párna

2, $R_x(\tau) \leq R_x(0)$ max v 0

3, $R_x(\tau) \xrightarrow{FT} ESD$

4, $R_x(0) = E_x$

Výkonové sig.

$$R_x(\tau) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,x(t+\tau)\,dt$$

Vlastnosti:

1, $R_x(\tau) = R_x(-\tau)$

2, $R_x(\tau) \leq R_x(0)$

3, $R_x(\tau) \xrightarrow{FT} PSD$

4, $R_x(0) = P_x$

Pr: Zistite, ktorá z nasledujúcich f. môže byť autokorelačná.

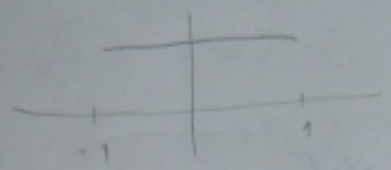
a) $R_x(\tau) = \begin{cases} 1 & -1 \le \tau \le 1 \\ 0 & \text{inak} \end{cases}$

b) $R_x(\tau) = \delta(\tau) + \sin 2\pi f_0 \tau$

c) $R_x(\tau) = e^{|\tau|}$

d) $R_x(\tau) = \begin{cases} 1-|\tau| ; & -1 \le \tau \le 1 \\ 0 & \text{inak} \end{cases}$

a)



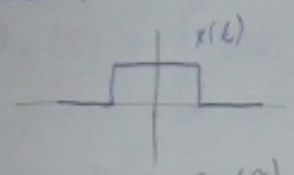

- energ.

1) platí
2) platí
3) neplatí

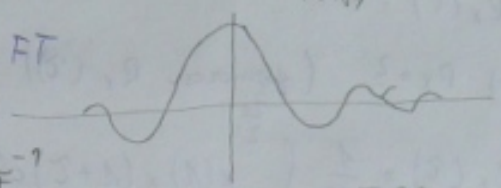
$R_x(\tau) \xrightarrow{FT}$ le 2 sine f2

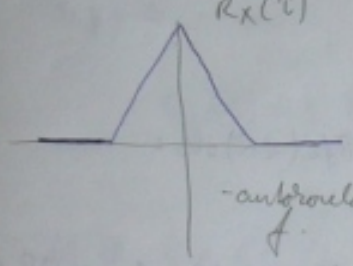
b) nie je párna, z $x(t)$ spravíme autokorelačnú jednoznačne, no naopak to neplatí

c) perm.:

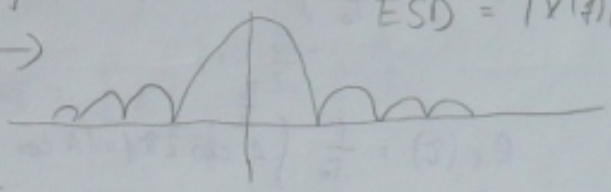



$\rightarrow$ FT




$\mathcal{F}^{-1}$ $\rightarrow$

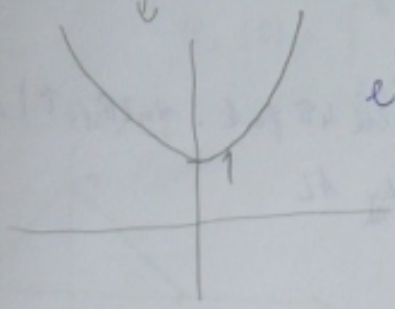


$ESD = |X(f)|^2 = \Psi_x(f)$

- autokorelač. f.

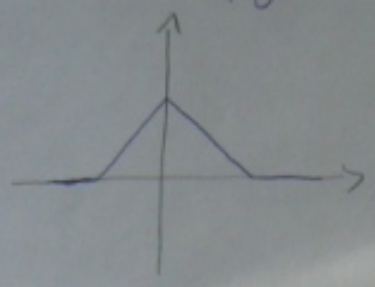
$\downarrow \mathcal{F}^{-1}$

c)




$e^{|\tau|}$ nespĺňa 2

d) $R_x(\tau) = \begin{cases} 1-|\tau| & -1 \le \tau \le 1 \\ 0 & \text{inak} \end{cases}$

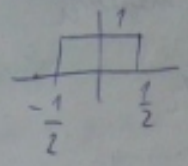


1) platí

2) platí

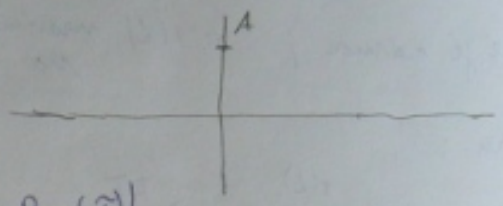
3) $R_x(\tau) \xrightarrow{FT} ESD$

4) $X(t)$



$\Rightarrow E_x = 1 \Rightarrow$ platí

Pr: $x(t) = A\cos 2\pi f_0 t$



a) $R_x(\tau) = ?$

b) $P_x = ?$ (pomocou $R_x(\tau)$)

$$R_x(\tau) = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\,x(t+\tau)\,dt$$

$$R_x(\tau) = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A\cos 2\pi f_0 t\, A\cos 2\pi f_0 (t+\tau)\,dt =$$

$$= \frac{A^2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos 2\pi f_0 t \cdot \{\cos 2\pi f_0 t \cdot \cos 2\pi f_0 \tau - \sin 2\pi f_0 t \cdot \sin 2\pi f_0 \tau\}\,dt =$$

$$= \frac{A^2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (\cos^2 2\pi f_0 t \cdot \cos 2\pi f_0 \tau - \sin 4\pi f_0 t \cdot \sin 2\pi f_0 \tau)\,dt =$$

vrátaly $A^2$

$$= \frac{A^2}{2}\cos 2\pi f_0 \tau + 0$$

**Př:** $P_x = R_x(0)$

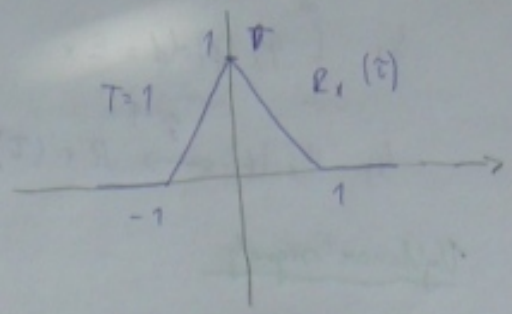
$$P_x = \frac{A^2}{2}$$

**Př:** $X(f) = \operatorname{sinc} f$

najděte $R_x(\tau)$ ex:

a) ESD

b) časová oblast

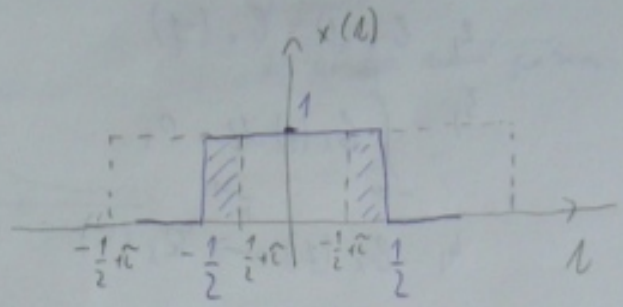



a) $\Psi_x(f) = |X(f)|^2 = \operatorname{sinc}^2 f$

$$\left.\begin{array}{l} 1 - \frac{|\tau|}{T} \; ; \; |\tau| \le T \\ 0 \; ; \; |\tau| > T \end{array}\right\} = R_x(\tau)$$

b) $X(f) = \operatorname{sinc} f$ vztah 12

$x(t) = \operatorname{rect}\left(\frac{t}{T}\right)$ $T = 1$



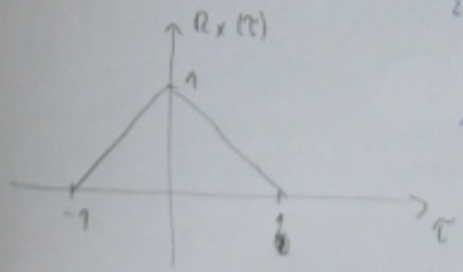

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, dt$$

a) $\tau \in \langle -1; 0 \rangle$

$$R_x(\tau) = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1\, dt = \left[ t \right]_{-\frac{1}{2}}^{\frac{1}{2}+\tau} = \tau + 1$$

b) $\tau \in \langle 0, 1 \rangle$

$$R_x(\tau) = \int_{-\frac{1}{2}+\tau}^{\frac{1}{2}} 1\, dt = \left[ t \right]_{-\frac{1}{2}+\tau}^{\frac{1}{2}} = 1 - \tau$$




vztah 15

# Vlastnosti spektrálních hustot

## Energ. signály:

1) $\Psi_x(f) \geq 0$ nezápor.

2) $\Psi_x(f) = \Psi_x(-f)$

3) $\int_{-\infty}^{\infty} \Psi_x(f)\,df = E_x$

4) $\Psi_x(f) \circ\!\!-\!\!\xrightarrow{\mathcal{F}^{-1}} R_x(\tau)$

## Výkonové signály:

1) $G_x(f) \geq 0$

2) $G_x(f) = G_x(-f)$

3) $\int_{-\infty}^{\infty} G_x(f)\,df = P_x$

4) $G_x(f) \circ\!\!-\!\!\xrightarrow{F^{-1}} R_x(\tau)$

Př: dvojstr. ESD $= \Psi_x(f) = 10^{-6} f^2$

Vypočtěte $E_x$ v pásmu $0 \div 10$ kHz

$$E_x = 2\int_{0}^{10^3} 10^{-6} f^2\,df = 2 \cdot 10^{-6} \left[\frac{f^3}{3}\right]_0^{10^4} = 2 \cdot 10^{-6} \cdot \frac{10^{12}}{3} = 2 \cdot \frac{10^{6}}{3} = \frac{2}{3}\ \mathrm{MJ}$$

Př: Zjistěte, zda a nakol. funkcí může být PSD reálného signálu.

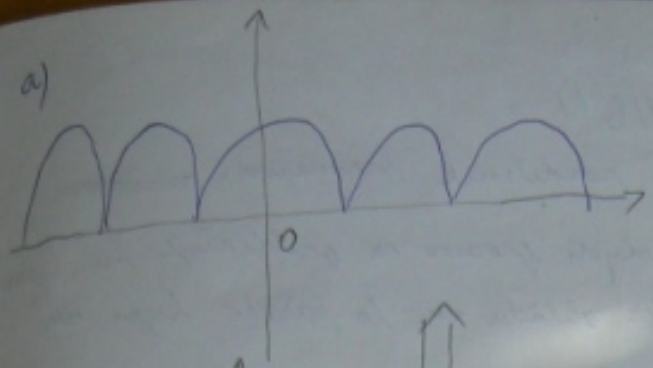
a) $G_x(f) = \delta(f) + \cos^2 2\pi f$

b) $G_x(f) = 10 + \delta(f-10)$

c) $G_x(f) = e^{-2\pi |f-10|}$

d) $G_x(f) = e^{-2\pi \cdot (f^2-10)}$

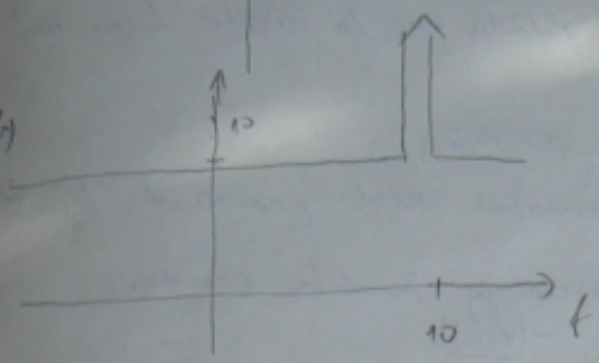
a)




a) platí

$$\int_{-\infty}^{\infty} G_x(f)\,df = \infty \neq P_x \text{ neplatí}$$

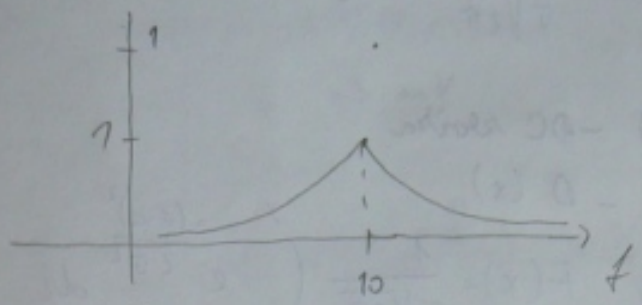
b)




a) platí

b) neplatí

c)

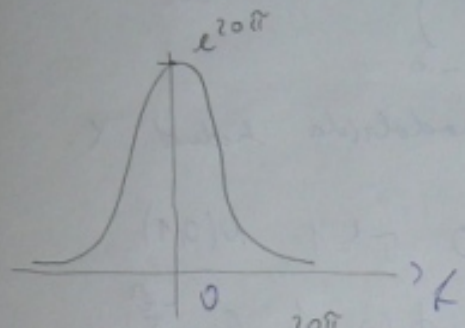


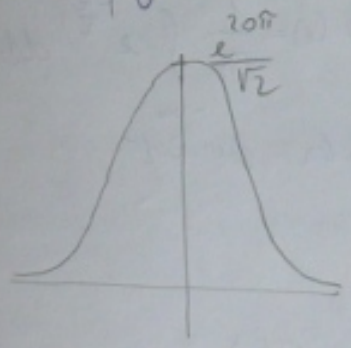

a) platí

b) ~~platí~~ není známa

c)

d)




a) ne.

b) ne.

c) ne.

d) $e^{\left[-\pi\left(\frac{f}{T}\right)^2\right]}$ - platí

PSD

# Cvičení 3.

## AWGN

Centrálna limitná veta – rozdelenie pravdepod. súčtu ~~súčtu~~ súčtu štatisticky nezávislých procesov sa približuje ku Gauss rozdeleniu ak $j \to \infty$ bez ohľadu na to, akého typu sú jednotlivé rozdelenia.

AWGN – má normálne rozd. pravdepod.

náh. premen. X má ~~sa~~ normálne rozd. pravdepod. $N(a, \sigma^2)$

s param. $a, \sigma^2$ $(\sigma > 0)$ ak jej hustota pravdep.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}}$$

a – str. $= E(x)$ – DC zložka

$\sigma^2$ – disperzia – $D(x)$

distribučná f $F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(t-a)^2}{2\sigma^2}} dt \to$

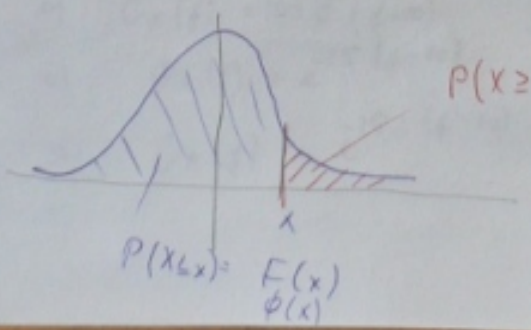
$\to$ vyjadruje pravdepod., že X nadobrala hodnot. X.

~~Normalit.~~ Normovaná normálne rozd $a = 0$ $\sigma^2 = 1$ $N(0,1)$

potom $f(x) = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{2}}$ $\phi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$ z tabuliky

ak $a \neq 0$ a $\sigma^2 \neq 1$ treba použiť

$$F(x) = \phi\left(\frac{x-a}{\sigma}\right)$$




$P(X \geq x) = Q(x)$

$Q(x)$ - vyjadřuje pravděp., že $X \leq N(0,1)$ nadobudne hodnotu $\geq x$

$$Q(x) = \frac{1}{\sqrt{x}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt$$

pre $N(0,\sigma^2)$: $\quad Q(x) = Q\left(\frac{x-a}{\sigma}\right)$

Poz:
platí: $Q(x) \quad Q(x) + \Phi(x) = 1$

$$\Phi(x) = 1 - \Phi(-x)$$

Pr: AWGN $\rightarrow \sigma_0^2 = 10^{-7}$. Vypočt. pravd., že napäťová úroveň šumu bude v int.

a) $+100\mu V \div +500\mu V$

b) $+1mV \div +3mV$

a) $N(0,\sigma^2)$ $\quad Q\left(\frac{100\mu V - 0}{\sqrt{10^{-7}}}\right) = Q\left(\frac{100 \cdot 10^{-6}}{\sqrt{10^{-7}}}\right) Q(0,316) =$ 0,32

$\sigma^2 = 10^{-7}$

$= P_1 = 0,3745$

$$Q\left(\frac{500 \cdot 10^{-6}}{\sqrt{10^{-7}}}\right) = Q(1,58) = 0,0571 = P_2$$

$P_{(100\mu V \leq x \leq 500\mu V)} = 0,3174$

b) $Q\left(\frac{1 \cdot 10^{-3}}{\sqrt{10^{-7}}}\right) = Q(3,16) = 0,0008$

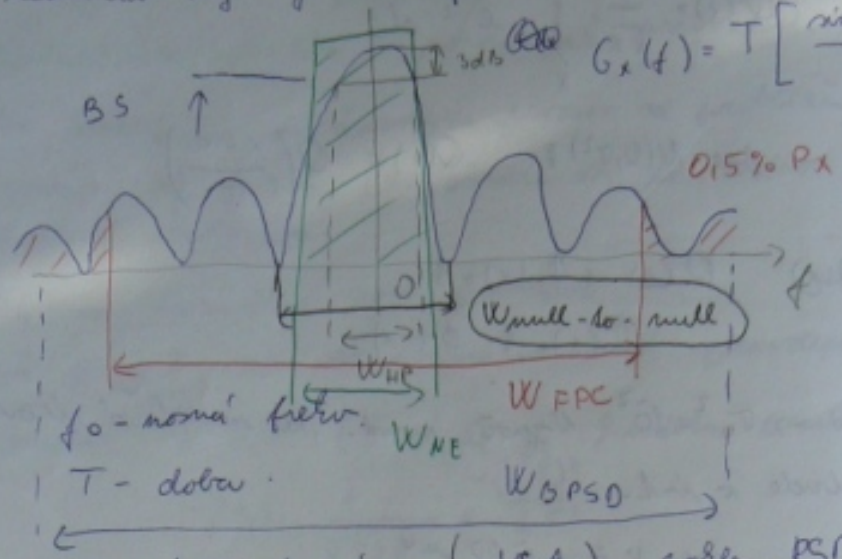
$$Q(9,48) = \left(\frac{1}{9,48 \cdot \sqrt{2\pi}}\right) \cdot e\left(-\frac{(9,48)^2}{2}\right) \doteq 0,042 \cdot 1,747 \cdot 10^{-10} = \doteq 1,203 \cdot 10^{-21}$$

$P\ (1mV \leq x \leq 3mV) = 0,0008$

# Frekvenčné pásmo

PSD reálneho sig. je $si^2$ funkcia

$$G_x(f) = T\left[\frac{\sin[\pi(f-f_0)T]}{\pi(f-f_0)T}\right]^2$$




$f_0$ - nosná frekv.

T - dobu

1, $W_{HP}$ - polovičné pásmo (v VSA) - podľa PSD o 3dB

HP - half power

2, $W_{NE}$ - pásmo, v kt. má AWGN rovnaký výkon ako je celkový výkon sig.

NE - noise equivalent

$$W_{NE} = \frac{P_x}{\max\{G_x(f)\}} \quad \Leftarrow \quad P_x = W_{NE} \cdot \max\{G_x(f)\}$$

3, $W_{null\text{-}to\text{-}null}$ - pásmo od 1. nuly po 2. nulu okr. max. výkonu (najpouž. definícia pre dig. kanál)

4, $W_{FPC}$ - pásmo, v kt. je 99% výkonu signálu

FPC - fractional power containment

5, Si $W_{BPSD}$ - mimo tohto pásma nesmie prekročiť danú hodnotu (BS alebo 50 dB

BPSD - bounded PSD)

6, Absolútne pásmo - pásmo, v kt. sme iste schopní merať nenulu $G_x(f)$

Pr: $G_x(f) = 10^{-4} \left\{ \frac{\sin[\pi(|f|-10^6).10^{-4}]}{\pi.(|f|-10^6).10^{-4}} \right\}^2$

vyp: $W_{HP}$

$W_{NE}$

$W_{null\ to\ null}$

$W_{FPC}$

$W_{BPSO}$

pre $f>0$: $G_x(f) = 10^{-4}\, si^2(\pi(f-10^6).10^{-4})$ l'horizontal

$f_c = 10^6\,Hz \rightarrow \max(G_x(f)) = 10^{-4}$

a) $W_{HP}$

$$10 \log \frac{G_x(10^6)}{G_x(f)} = 3\,dB \Rightarrow G_x(f) = \frac{1}{2} G_x(10^6)$$

$f(x)$

$$10^{-4}\, si^2(\pi(f'-10^6).10^{-4}) = \frac{1}{2}.10^{-4}$$

$$\left(\frac{\sin[\pi(f'-10^6).10^{-4}]}{\pi.(f'-10^6).10^{-4}}\right)^2 = \frac{1}{2}$$

subst. $\pi.(f'-10^6).10^{-4} = L$

$$\left(\frac{\sin L}{L}\right)^2 = \frac{1}{2} \Rightarrow \sin L = \pm \frac{1}{\sqrt{2}}.L$$

(len numerické riešenie)

iterácia

$L_0 = 1$

$L_i = \sin(L_{i+1}).\sqrt{2}$

$L_1 = 0{,}0247$

$L_2 = 6{,}096.10^{-4}$

$L = \pm 1{,}39156$

$\pi.(f'-10^6).10^{-4} = \pm 1{,}39156$

$f' = 10^6 \pm 4{,}43.10^3 \Rightarrow f_H = 10^6 + 4{,}43.10^3$

$f_D = 10^6 - 4{,}43.10^3$

$W_{HP} = f_H - f_D = 8{,}86\,kHz$

b) $W_{NE}$

$W_{NE} = \frac{P_x}{\max G_x(f)}$ ; $R_x(0) = P_x$

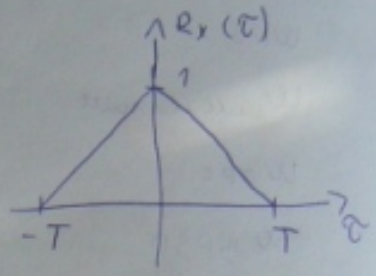
$R_x(\tau) \rightarrow G_x(f)$

$$T \operatorname{sinc}^2 f T \xrightarrow{\mathcal{F}^{-1}} \begin{cases} 1 - \frac{|\tau|}{T} & |\tau| \le T \\ 0 & |\tau| > T \end{cases}$$

$T = 10^{-4}$

$R_x(0) = P_x = 1 =$

$\Rightarrow W_{NE} = \frac{1}{10^{-4}} = 10\,kHz$




c) $W_{null-to-null}$ : $G_x(f) = 0$

$10^{-4} \sin^2 L = 0$

$10^{-4} \sin^2 L = 0 \Rightarrow L = 0 + k\pi$ , $k = 1, 2 \ldots$

1. nula $\Rightarrow k = 1$

$\pi \cdot (f^2 - 10^6) \cdot 10^{-4} = \pi$

$f^2 - 10^6 = 10^4$

$$\left.\begin{aligned} f^2 - 10^6 = 10^4 \Rightarrow f_H &= 10^6 + 10^4 \\ f_D &= 10^6 - 10^4 \end{aligned}\right\} \Rightarrow$$

$\Rightarrow W_{null-to-null} = 20\,kHz$

d) $W_{FPC} = 205{,}72\,kHz$

e) $W_{BPSD} = 370\,kHz$