Metóda "per partes" pre neurčité integrály

Ak funkcie $f, g \subseteq R \times R$ majú na otvorenom intervale $\mathcal{J}$ spojité derivácie f', g' potom

$$\int f'(x) g(x)\,dx = f(x) g(x) - \int f(x) \cdot g'(x)\,dx, \text{ na } \mathcal{J}$$

Na intervale $\mathcal{J} = (0, \infty)$ je:

$$\int \ln x\,dx = \int \underbrace{1}_{f'(x)} \cdot \underbrace{\ln x}_{g(x)}\,dx = \underbrace{x}_{f(x)} \underbrace{\ln x}_{g(x)} - \int \underbrace{x}_{f(x)} \underbrace{\frac{1}{x}}_{g'(x)}\,dx =$$

$$= x \ln x - \int dx = x \ln x - x + C = \underline{x(\ln x - 1) + C}$$

Na intervale $\mathcal{J} = (0, \infty)$ je

$$\int \underbrace{x}_{f'(x)} \underbrace{\ln x}_{g(x)}\,dx = \underbrace{\frac{x^2}{2}}_{f(x)} \underbrace{\ln x}_{g(x)} - \int \underbrace{\frac{x^2}{2}}_{f(x)} \cdot \underbrace{\frac{1}{x}}_{g'(x)}\,dx =$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x\,dx = \underbrace{\frac{x^2}{2} \ln x - \frac{1}{2} \frac{x^2}{2} + C}_{\text{pre } x \in (0,\infty)}$$

$$\int \underbrace{x}_{f_1'(x)} \underbrace{\ln^2 x}_{g_1(x)}\,dx = \underbrace{\frac{x^2}{2}}_{f_1(x)} \underbrace{\ln^2 x}_{f_2(x)} - \int \underbrace{\frac{x^2}{2}}_{f_1(x)} \underbrace{2(\ln x) \frac{1}{x}}_{f_2'(x)}\,dx =$$

$$= \frac{x^2}{2} \cdot \ln^2 x - \int \underbrace{x}_{f_2'(x)} \underbrace{\ln x}_{g_2(x)}\,dx = \frac{x^2}{2} \ln^2 x - \Big\{ \underbrace{\frac{x^2}{2}}_{f_2(x)} \underbrace{\ln x}_{g_2(x)} -$$

opakujeme "per partes"

$$- \int \underbrace{\frac{x^2}{2}}_{f_2(x)} \cdot \underbrace{\frac{1}{x}}_{g_2'(x)}\,dx \Big\} = \underbrace{\frac{x^2}{2} \ln^2 x - \frac{x^2}{2} \ln x - \frac{x^2}{4} + C}_{\text{na intervale } \mathcal{J} = (0,\infty)}$$

$$\int \operatorname{arctg} x \, dx = \int \underbrace{1}_{f'(x)} \cdot \underbrace{\operatorname{arctg} x}_{g(x)} \, dx = \underbrace{x}_{f(x)} \, \underbrace{\operatorname{arctg} x}_{g(x)} -$$

$$- \int \underbrace{x}_{f(x)} \underbrace{\frac{1}{x^2+1}}_{g'(x)} \, dx = x \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{x^2+1} \, dx =$$

$$= x \operatorname{arctg} x - \frac{1}{2} \ln (x^2+1)$$ na intervale $J=(-\infty,\infty)$

$$\int \underbrace{x}_{f'(x)} \, \underbrace{\operatorname{arctg} x}_{g(x)} \, dx = \underbrace{\frac{x^2}{2}}_{f(x)} \, \underbrace{\operatorname{arctg} x}_{g(x)} - \int \underbrace{\frac{x^2}{2}}_{f(x)} \, \underbrace{\frac{1}{x^2+1}}_{g'(x)} \, dx =$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2}{x^2+1} \, dx$$

vsuvka:

$$\int \frac{x^2}{x^2+1} \, dx = \int \frac{(x^2+1)-1}{x^2+1} \, dx = \int \left(1 - \frac{1}{x^2+1}\right) dx =$$

$$= x - \operatorname{arctg} x + C$$ na $J=(-\infty,\infty)$

$$\int x \operatorname{artg} x = \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} (x - \operatorname{artg} x) + C$$

na $J=(-\infty,\infty)$

$$\int \underbrace{x^3}_{f_1(x)} \underbrace{e^x}_{g_1'(x)} \, dx = \underbrace{x^3}_{f_1(x)} \underbrace{e^x}_{g_1(x)} - \int \underbrace{\underbrace{3x^2}_{f_1'(x)}}_{f_2(x)} \underbrace{\underbrace{e^x}_{g_1(x)}}_{g_2'(x)} \, dx =$$ → opakujeme per partes

$$= x^3 e^x - \Big\{ \underbrace{3x^2}_{f_2(x)} \underbrace{e^x}_{g_2(x)} - \int \underbrace{\underbrace{6x}_{f_2'(x)}}_{f_3(x)} \underbrace{\underbrace{e^x}_{g_2(x)}}_{g_3'(x)} \, dx \Big\} =$$ → opakujeme per partes

$$= x^3 e^x - 3x^2 e^x + 6x e^x - \int \underbrace{6}_{f_3'} \underbrace{e^x}_{g_3} \, dx =$$

$$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$$ na $J=(-\infty,\infty)$

Metóda "per partes" pre určité integrály

Ak funkcie $f, g \subseteq R \times R$ majú spojité derivácie na $\langle a, b \rangle$, potom

$$\int_a^b f'(x)g(x)\,dx = \left[f(x)g(x)\right]_a^b - \int_a^b f(x)\,g'(x)\,dx$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{x^2}_{f_1(x)} \underbrace{\sin x}_{g_1'(x)}\,dx = \Big[\underbrace{x^2}_{f_1(x)}\underbrace{(-\cos x)}_{g_1(x)}\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{2x}_{f_1'(x)}\underbrace{(-\cos x)}_{g_1(x)}\,dx =$$

$$= \frac{\pi^2}{4}\underbrace{\left(-\cos\frac{\pi}{2}\right)}_{0} - \frac{\pi^2}{4}\underbrace{\left(-\cos\left(-\frac{\pi}{2}\right)\right)}_{0} + 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{x}_{f_2(x)}\underbrace{\cos x}_{g_2'(x)}\,dx =$$

– opakujeme per partes

$$= 2\left\{\Big[\underbrace{x}_{f_2(x)}\underbrace{\sin x}_{g_2(x)}\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{1}_{f_2'(x)}\cdot\underbrace{\sin x}_{g_2(x)}\,dx\right\} =$$

$$= 2\left\{\left[\frac{\pi}{2} + \left(-\frac{\pi}{2}\right)\right) - \left[-\cos x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right\} =$$

$$= 2\{0 + 0 - 0\} = \underline{\underline{0}}$$

$$\int_1^e \underbrace{\frac{1}{x}}_{f'(x)}\cdot\underbrace{\ln x}_{g(x)}\,dx = \Big[\underbrace{\ln^2 x}_{f(x)\cdot g(x)}\Big]_1^e - \int_1^e \underbrace{\ln x}_{f(x)}\cdot\underbrace{\frac{1}{x}}_{g'(x)}\,dx$$

z toho:

$$2\int_1^e \frac{1}{x}\ln x\,dx = \underbrace{\ln^2 e}_{1} - \underbrace{\ln^2 1}_{0} = 1$$

$$\int_1^e \frac{1}{x}\ln x\,dx = \frac{1}{2}$$

$$\int_1^e \underbrace{\frac{1}{x^2}}_{f'(x)} \underbrace{\ln x}_{g(x)} \, dx = \Big[\underbrace{-\frac{1}{x}}_{f(x)} \underbrace{\ln x}_{g(x)}\Big]_1^e - \int_1^e \underbrace{\left(-\frac{1}{x}\right)}_{f(x)} \underbrace{\frac{1}{x}}_{g'(x)} \, dx =$$

$$= -\frac{1}{e} \underbrace{\ln e}_{=1} + 1 \underbrace{\ln 1}_{=0} + \int_1^e x^{-2} \, dx = -\frac{1}{e} +$$

$$+ \left[\frac{x^{-1}}{-1}\right]_1^e = -\frac{1}{e} - \frac{1}{e} + 1 = 1 - \frac{2}{e}$$

$$\int \sin^2 x \, dx = \int \underbrace{\sin x}_{f'(x)} \cdot \underbrace{\sin x}_{g(x)} \, dx =$$

$$= (-\cos x) \sin x - \int (-\cos x) \cdot \cos x \, dx = -\sin x \cos x +$$

$$+ \int \underbrace{\cos^2 x}_{= 1 - \sin^2 x} \, dx = -\sin x \cos x + \int dx - \int \sin^2 x \, dx$$

$= \frac{\sin 2x}{2}$

z toho: $$2 \int \sin^2 x \, dx = -\sin x \cos x + x \quad / \cdot \frac{1}{2}$$

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

iný spôsob:

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$, pretože

$$\cos 2x = \cos^2 x - \sin^2 x$$

a teda $$1 - \cos 2x = 1 - \cos^2 x + \sin^2 x = 2 \sin^2 x$$

$$\int \sin^2 x \, dx = \int \left(\frac{1}{2} - \frac{1}{2} \cos 2x\right) dx =$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C$$

$$\int_0^{2\pi} \sin^2 x \, dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{2\pi} = \frac{2\pi}{2} = \pi$$