Derivácia funkcie v jednotkovom smere

Nech $f \subseteq R^n \times R$ je funkcia a nech $\bar{a} \in R^n$ je vnútorným bodom $D(f)$.
Nech $\bar{u} \in R^n$ je vektor a $\|\bar{u}\| = 1$.
Potom ak $\bar{a} = (a_1, a_2, \dots, a_n)$, $\bar{u} = (u_1, u_2, \dots, u_n)$ a existuje číslo

$$\boxed{D_{\bar{u}} f(\bar{a})} = \lim_{h \to 0} \frac{f(\bar{a} + h\bar{u}) - f(\bar{a})}{h} =$$

$$= \lim_{h \to 0} \frac{f(a_1 + hu_1, a_2 + hu_2, \dots, a_n + hu_n) - f(a_1, \dots, a_n)}{h}$$

tak ho nazývame derivácion f v bode $\bar{a}$ v jednotkovom smere $\bar{u}$.

- Pomocou reťazového pravidla dokážte:

Ak funkcia $f \subseteq R^n \times R$ je diferencovateľná v bode $\bar{a} = (a_1, a_2, \dots, a_n)$, potom $D_{\bar{u}} f(\bar{a})$ existuje v každom jednotkovom smere $\bar{u} = (u_1, u_2, \dots, u_n) \in R^n$ a $\boxed{D_{\bar{u}} f(\bar{a}) = \nabla f(\bar{a}) \cdot \bar{u}}$ skalárny súčin

Riešenie: Pre dané $\bar{a}$ a $\bar{u}$ je funkcia $F(h) = f(a_1 + hu_1, a_2 + hu_2, \dots, a_n + hu_n)$ funkciou jednej premennej h a podľa reťazového pravidla (2) je: (zrejme $D_{\bar{u}} f(\bar{a}) = [F'(h)]_{h=0}$)

$$D_{\bar{u}}f(\bar{a})=\left[\frac{dF(h)}{dh}\right]_{h=0}=\left[\frac{\partial f(x_1,x_2,\dots,x_n)}{\partial x_1}\right]_{\bar{x}=\bar{a}}\cdot\left[\frac{d(a_1+hu_1)}{dh}\right]_{h=0}+$$

$$+\left[\frac{\partial f(x_1,x_2,\dots,x_n)}{\partial x_2}\right]_{\bar{x}=\bar{a}}\cdot\left[\frac{d(a_2+hu_2)}{dh}\right]_{h=0}+\dots+$$

$$+\left[\frac{\partial f(x_1,x_2,\dots,x_n)}{\partial x_n}\right]_{\bar{x}=\bar{a}}\cdot\left[\frac{d(a_n+hu_n)}{dh}\right]_{h=0}=$$

$$=\nabla f(\bar{a})\cdot\bar{u}=D_{\bar{u}}f(\bar{a})$$

pretože $\left[\frac{d(a_k+hu_k)}{dh}\right]_{h=0}=u_k$ pre $k=1,2,\dots,n$

a teda $\left[\frac{dF(h)}{dh}\right]_{h=0}=\left(\left[\frac{\partial f(\bar{x})}{\partial x_1}\right]_{\bar{x}=\bar{a}},\dots,\left[\frac{\partial f(\bar{x})}{\partial x_n}\right]_{\bar{x}=\bar{a}}\right)\cdot(u_1,\dots,u_n)$

kde $x_1=a_1+hu_1,\ x_2=a_2+hu_2,\dots,x_n=a_n+hu_n$ | skalárny súčin

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- Ak funkcia $f\subseteq R^n\times R$ je diferencovateľná v bode $\bar{a}\in D(f)$ $(O_\delta(\bar{a})\subseteq D(f))$ nájdite $D_{\bar{e}_k}f(\bar{a})$, kde $\bar{e}_k=(0,0,\dots,1,0,\dots,0)$ pre $k=1,2,\dots,n$

k-tá ~~zl~~ zložka

Riešenie: $D_{\bar{e}_k}f(\bar{a})=\nabla f(\bar{a})\cdot\bar{e}_k=$

$$=\left(\left[\frac{\partial f(\bar{x})}{\partial x_1}\right]_{\bar{x}=\bar{a}},\dots,\left[\frac{\partial f(\bar{x})}{\partial x_k}\right]_{\bar{x}=\bar{a}},\dots,\left[\frac{\partial f(\bar{x})}{\partial x_n}\right]_{\bar{x}=\bar{a}}\right)\cdot(0,0,\dots,1,\dots,0)$$

$$=\left[\frac{\partial f(\bar{x})}{\partial x_k}\right]_{\bar{x}=\bar{a}}$$

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$f(x,y,z) = 6 - 3x^2 - y^2 - z$

$\bar{a} = (1,1,2) \qquad \bar{u} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

nájdime $\nabla f(\bar{a})$, $D_{\bar{u}} f(\bar{a})$, $Df_{\bar{a}}(\bar{x})$

$\frac{\partial f}{\partial x} = -6x$, $\frac{\partial f}{\partial y} = -2y$, $\frac{\partial f}{\partial z} = -1$ a sú spojité na R^3 $\Longrightarrow$ f je diferencovateľná v každom bode z R^3.

Pre dané $\bar{a} = (1,1,2)$ je $\left[\frac{\partial f}{\partial x}\right]_{\bar{x}=\bar{a}} = [-6x]_{\substack{x=1\\y=1\\z=2}} = -6$,

$\left[\frac{\partial f}{\partial y}\right]_{\bar{x}=\bar{a}} = [-2y]_{\substack{x=1\\y=1\\z=2}} = -2$, $\left[\frac{\partial f}{\partial z}\right]_{\bar{x}=\bar{a}} = [-1]_{\substack{x=1\\y=1\\z=2}} = -1$

$\nabla f(\bar{a}) = (-6,-2,-1)$

$D_{\bar{u}} f(\bar{a}) = \nabla f(\bar{a}) \circ \bar{u} = (-6,-2,-1) \circ \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) =$

$= -\frac{6}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$

$Df_{\bar{a}}(\bar{x}) = \nabla f(\bar{a}) \circ (\bar{x} - \bar{a}) = (-6,-2,-1) \circ (x-1, y-1, z-2) =$

$= -6(x-1) - 2(y-1) - (z-2)$

Teda napr. dotyková rovina k ploche $6 - 3x^2 - y^2 - z = 0$ v bode $\bar{a} = (1,1,2)$ má rovnicu

$6(x-1) + 2(y-1) + (z-2) = 0$

$6x + 2y + z = 10$