Gradient funkcie $f \subseteq R^n \times R$ a dotyková rovina k ploche

ak pre funkciu $f \subseteq R^n \times R$ existujú $\left[\frac{\partial f(\bar{x})}{\partial x_k}\right]_{\bar{x}=\bar{a}}$

potom vektor $\nabla f(\bar{a}) = \left(\left[\frac{\partial f(\bar{x})}{\partial x_1}\right]_{\bar{x}=a}, \dots, \left[\frac{\partial f(\bar{x})}{\partial x_n}\right]_{\bar{x}=\bar{a}}\right)$

nazývame **gradientom** f v bode $\bar{a} \in R^n$.

- $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $\bar{a} = (a_1, a_2, a_3) \in R^3$, $\bar{a} \neq \bar{0}$

potom
$$\left[\frac{\partial f(x,y,z)}{\partial x}\right]_{\bar{x}=\bar{a}} = \left[\frac{x}{\sqrt{x^2+y^2+z^2}}\right]_{\substack{x=a_1\\ y=a_2\\ z=a_3}} = \frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}}$$

$$\left[\frac{\partial f(x,y,z)}{\partial y}\right]_{\bar{x}=\bar{a}} = \left[\frac{y}{\sqrt{x^2+y^2+z^2}}\right]_{\substack{x=a_1\\ y=a_2\\ z=a_3}} = \frac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}}$$

$$\left[\frac{\partial f(x,y,z)}{\partial z}\right]_{\bar{x}=\bar{a}} = \left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right]_{\substack{x=a_1\\ y=a_2\\ z=a_3}} = \frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}$$

$$\nabla f(\bar{a}) = \left(\frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}}, \frac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}}, \frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}\right)$$

v každom bode $\bar{a} \neq (0,0,0)$.

napr. $\nabla f((1,1,1)) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\nabla f((0,0,1)) = \left(0, 0, \frac{1}{\sqrt{3}}\right)$, $\nabla f((0,1,0)) = \left(0, \frac{1}{\sqrt{3}}, 0\right)$

ak pre plochu $f(x,y,z)=c$ (v R^3) je funkcia $f\subseteq R^3\times R$ diferencovateľná v bode $\bar{a}=(a_1,a_2,a_3)\in R^3$, potom $\nabla f(\bar{a})$ je vektor normály dotykovej roviny k ploche $f(x,y,z)=c$ v bode $\bar{a}$. Teda ak $\bar{x}$ je ľubovoľný bod tej dotykovej roviny tak $\underbrace{\nabla f(\bar{a})\circ(\bar{x}-\bar{a})=0}_{\circ \text{ skalárny súčin v } R^3}$ je rovnica tej dotykovej roviny ϱ k ploche $f(x,y,z)=c$ v bode $\bar{a}$.

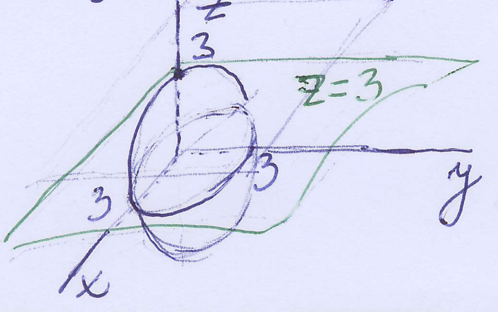
Teda
$$\varrho\equiv \underbrace{\left[\frac{\partial f(\bar{x})}{\partial x}\right]_{\bar{x}=\bar{a}}(x_1-a_1)+\left[\frac{\partial f(\bar{x})}{\partial y}\right]_{\bar{x}=\bar{a}}(x_2-a_2)+\left[\frac{\partial f(\bar{x})}{\partial z}\right]_{\bar{x}=\bar{a}}(z-a_3)=0}$$

$$\varrho\equiv \quad Df_{\bar{a}}(\bar{x})=0$$

- Dotyková rovina ku guľovej ploche $\underbrace{x^2+y^2+z^2}_{f(x,y,z)}=9$ v bode $\bar{a}=(0,0,3)$ je:

$$\left[\frac{\partial f(x,y,z)}{\partial x}\right]_{\bar{x}=\bar{a}}(x-0)+\left[\frac{\partial f(x,y,z)}{\partial y}\right]_{\bar{x}=\bar{a}}(y-0)+\left[\frac{\partial f(x,y,z)}{\partial z}\right]_{\bar{x}=\bar{a}}(z-3)=0$$

$$\left[2x\right]_{\substack{x=0\\y=0\\z=3}}(x-0)+\left[2y\right]_{\substack{x=0\\y=0\\z=3}}(y-0)+\left[2z\right]_{\substack{x=0\\y=0\\z=3}}(z-3)=0$$



$$6(z-3)=0$$

je rovnica tej dotyk. rov. teda $\boxed{\varrho\equiv z=3}$

Dotyková rovina ku grafu funkcie

$z = g(x,y)$ (ak g je diferenc. v bode (a_1,a_2)) je vlastne dotyková rovina k ploche

$$\underbrace{g(x,y) - z = 0}_{f(x,y,z)=0}$$

v bode $(a_1, a_2, \underbrace{g(a_1,a_2)}_{a_3 = g(a_1,a_2)})$

je bod grafu f

a tá dotyková rovina má rovnicu:

$$\left[\frac{\partial g(x,y)}{\partial x}\right]_{\substack{x=a_1\\y=a_2}}(x-a_1) + \left[\frac{\partial g(x,y)}{\partial y}\right]_{\substack{x=a_1\\y=a_2}}(y-a_2) - (z-a_3) = 0$$

(pretože $\frac{\partial f(x,y,z)}{\partial z} = -1$)

• Najdime dotykovú rovinu ku grafu funkcie $g(x,y) = 6-3x^2-y^2$ v bode $(1,2,g(1,2))$.

Grafom funkcie g je plocha $z = 6-3x^2-y^2$

a teda: $\underbrace{3x^2+y^2+z}_{f(x,y,z)} = 6$, $\bar{a} = (1,2,-1)$

$$\rho \equiv \nabla f(\bar{a}) \cdot (\bar{x}-\bar{a}) = 0$$

$$\rho \equiv (6,4,1)\cdot(x-1, y-2, z+1) = 0$$

$$\rho \equiv 6(x-1)+4(y-2)+(z+1) = 0$$

$$\rho \equiv \boxed{6x+4y+z=13}$$

$$\left[\frac{\partial f(\bar{x})}{\partial x}\right]_{\bar{x}=\bar{a}} = [6x]_{\substack{x=1\\y=2\\z=-1}} = 6$$

$$\left[\frac{\partial f(\bar{x})}{\partial y}\right]_{\bar{x}=\bar{a}} = [2y]_{\substack{x=1\\y=2\\z=-1}} = 4$$

$$\left[\frac{\partial f(\bar{x})}{\partial z}\right]_{\bar{x}=\bar{a}} = 1$$

Najdite bod v ktorom je dotyková rovina ku grafu funkcie $g(x,y) = 6-3x^2-y^2$ rovnobežná s rovinou $3x-2y+z=3$ a rovnica tej dotykovej roviny.

grafom g je plocha $z=6-3x^2-y^2$ teda $3x^2+y^2+z=6$

$\nabla f(\bar{a}) = (6x, 2y, 1)_{\substack{x=a_1\\y=a_2\\z=a_3}} = (6a_1, 2a_2, 1)$ je

vektor normály hľadanej dotykovej roviny ϱ v hľadanom bode $\bar{a}=(a_1,a_2,a_3)$,

vektor normály danej roviny $3x-2y+z=3$ je $(3,-2,1)$ a teda bude platiť

$k.(3,-2,1) = (6a_1, 2a_2, 1) \Longleftrightarrow \left.\begin{array}{r} 3k = 6a_1 \\ -2k = 2a_2 \\ k = 1 \end{array}\right\} \Rightarrow$

$\Rightarrow 3 = 6a_1, -2 = 2a_2 \Longrightarrow a_1 = \frac{1}{2}, a_2 = -1, a_3 = g(a_1,a_2) = \frac{17}{4}$. Dosadením je $\nabla f(\bar{a}) = (6\cdot\frac{1}{2}, 2\cdot(-1), 1) =$

$= (3, -2, 1)$, $\bar{a} = (\frac{1}{2}, -1, \frac{17}{4})$

$\varrho \equiv \nabla f(\bar{a})\circ(\bar{x}-\bar{a}) = 0$ má teda tvar

$\varrho \equiv (3,-2,1)\circ(x-\frac{1}{2}, y+1, z-\frac{17}{4}) = 0$

$\varrho \equiv 3(x-\frac{1}{2}) - 2(y+1) + z - \frac{17}{4} = 0 \;/\cdot 2$

$\varrho \equiv 6x - 4y + 2z = \frac{31}{2}$

resp. $\varrho \equiv 3x - 2y + z = \frac{31}{4}$, dotykový bod $\bar{a} = (\frac{1}{2}, -1, \frac{17}{4})$