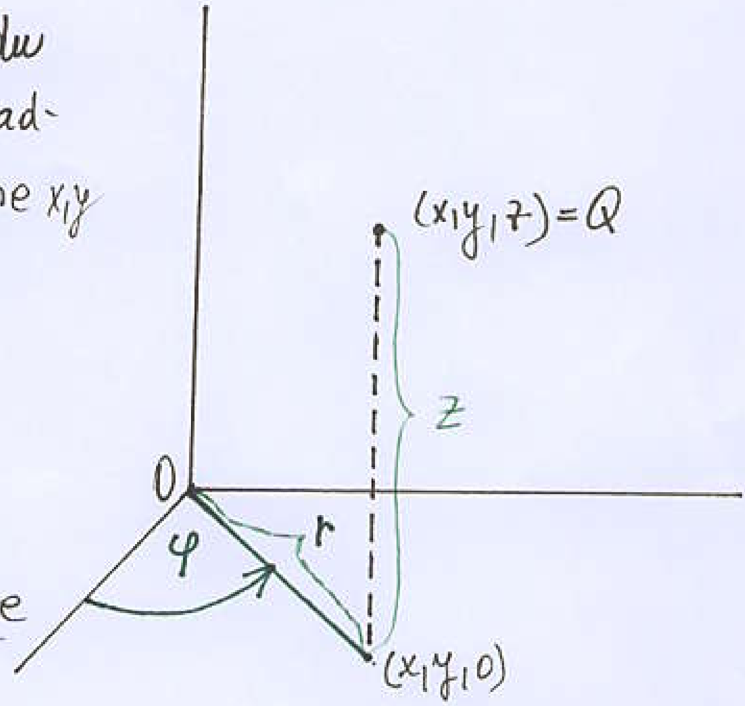
Transformácia trojných integrálov do cylindrických súradníc

cylindrické súradnice bodu $Q=(x,y,z)$ sú polárne súradnice bodu $(x,y,0)$ v rovine x,y a z-tová súradnica Q



teda: $r \geq 0$
$\varphi \in \langle 0, 2\pi)$
$z \in R$

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= z \end{aligned}\right\} \text{transf. rovnice}$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r}, & \frac{\partial x}{\partial \varphi}, & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r}, & \frac{\partial y}{\partial \varphi}, & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r}, & \frac{\partial z}{\partial \varphi}, & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\varphi, & -r\sin\varphi, & 0 \\ \sin\varphi, & r\cos\varphi, & 0 \\ 0, & 0, & 1 \end{vmatrix} =$$

$$= \begin{vmatrix} \cos\varphi, & -r\sin\varphi \\ \sin\varphi, & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = \underline{r}$$

Veta. Nech elementárna oblasť $A \subseteq R^3$ je v cylindr. súr. daná nerovnosťami: (typ $\langle\langle \varphi, r, z \rangle\rangle$)

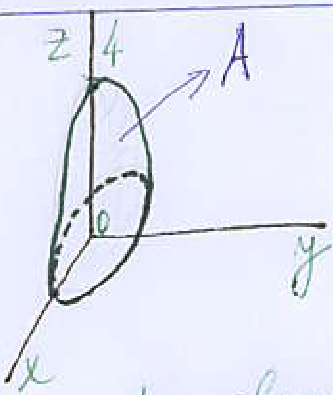
$$\left.\begin{aligned} \left.\begin{aligned} \alpha &\leq \varphi \leq \beta \\ h_1(\varphi) &\leq r \leq h_2(\varphi) \end{aligned}\right\} D \\ g_1(\varphi,r) \leq z \leq g_2(\varphi,r) \end{aligned}\right\} A$$

$h_1, h_2 \subseteq R\times R$ sú spojité na $\langle \alpha, \beta \rangle$
$g_1, g_2 \subseteq R^2\times R$ sú spojité na D.

Nech f je spojitá na A, potom

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \int_\alpha^\beta \left(\int_{h_1(\varphi)}^{h_2(\varphi)} \left(\int_{g_1(\varphi,r)}^{g_2(\varphi,r)} f(r\cos\varphi, r\sin\varphi, z)\, \overset{|J|}{r}\, dz \right) dr \right) d\varphi$$

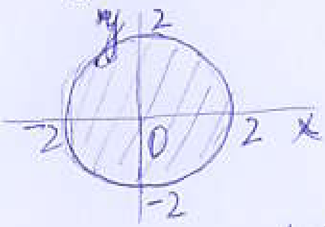
Vypočítajte objem telesa A ohraničeného rovinou xy a paraboloidom $z = 4 - x^2 - y^2$



základňa v rovine xy:

$$z = 0 \Rightarrow 0 = 4 - x^2 - y^2$$
$$x^2 + y^2 = 4$$

je kružnica o polomere 2:



v polárnych súr.:

$$0 \le r \le 2$$
$$0 \le \varphi \le 2\pi$$

transformačné rovnice:

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= z \end{aligned}\right\} (*) \quad |J| = r$$

z rovnice toho paraboloidu je: $z = 4 - x^2 - y^2$ súradnice

a teda pre body toho telesa pre z-tové súradnice

platí $0 \le z \le 4 - x^2 - y^2$

v cylindrických $0 \le z \le 4 - r^2\cos^2\varphi - r^2\sin^2\varphi$

Teda A je popísaná (v cylindr. súr.):
$$\left.\begin{aligned} 0 &\le r \le 2 \\ 0 &\le \varphi \le 2\pi \\ 0 &\le z \le 4 - r^2 \end{aligned}\right\} A$$

$$C(A) = \int_0^2 \left(\int_0^{2\pi} \left(\int_0^{4-r^2} \underbrace{r}_{|J|} \, dz \right) d\varphi \right) dr =$$

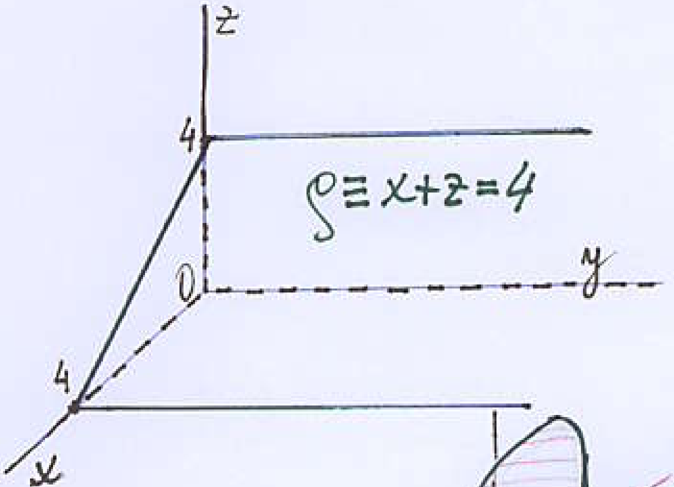
$$= \int_0^2 \left(\int_0^{2\pi} \left[r \cdot z \right]_0^{4-r^2} d\varphi \right) dr = \int_0^2 \left(\int_0^{2\pi} r(4-r^2)\, d\varphi \right) dr =$$

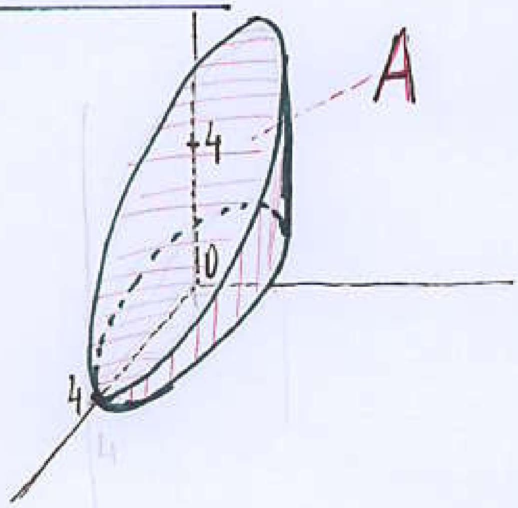
$$= \int_0^2 r(4-r^2) \left[\varphi \right]_0^{2\pi} dr = 2\pi \left[4\frac{r^2}{2} - \frac{r^4}{4} \right]_0^2 = 2\pi(8 - 4) = \underline{\underline{8\pi}}$$

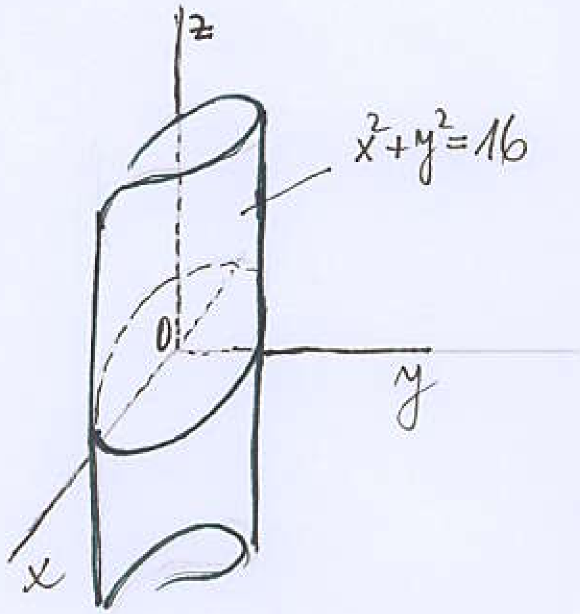
$$\iiint_A (x^2+y^2)\,dx\,dy\,dz = \int_0^2 \left(\int_0^{2\pi} \left(\int_0^{4-r^2} \underbrace{r}_{|J|} \cdot \underbrace{r^2}_{x^2+y^2=r^2} dz \right) d\varphi \right) dr = \int_0^2 \left(\int_0^{2\pi} r^3(4-r^2)\, d\varphi \right) dr =$$

$$= \int_0^{2\pi} d\varphi \int_0^2 (4r^3 - r^5)\, dr = 2\pi \left[r^4 - \frac{r^6}{6} \right]_0^2 = 2\pi\left(2^4 - \frac{2^5}{3}\right) = \frac{2^5}{3}\pi$$

- **A** je teleso ohraničené valcovou plochou $x^2+y^2=16$ a rovinami: $z=0$ a $x+z=4$







transf.:

$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= z \end{aligned}\right\} \quad |J| = r$$

$\varrho \equiv x+z=4$

$z = 4-x$

v cylindr. $z = 4 - r\cos\varphi$

$$A: \begin{cases} 0 \le r \le 4 \\ 0 \le \varphi \le 2\pi \\ 0 \le z \le 4 - r\cos\varphi \end{cases}$$

$$C(A) = \iiint\limits_A dx\,dy\,dz = \int_0^4 \left(\int_0^{2\pi} \left(\int_0^{4-r\cos\varphi} r \, dz \right) d\varphi \right) dr =$$

$$= \int_0^4 \left(\int_0^{2\pi} r(4 - r\cos\varphi)\, d\varphi \right) dr = \int_0^4 \left[4r\varphi - r^2 \sin\varphi \right]_0^{2\pi} dr =$$

$$= \int_0^4 4r\,2\pi\, dr = 8\pi \left[\frac{r^2}{2} \right]_0^4 = 64\pi$$

Vypočítajte $\iiint\limits_A \sqrt{x^2+y^2}\,dx\,dy\,dz$, kde

A je teleso zo strany 20.

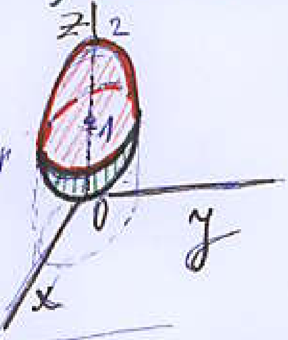
Teda transf.: $\left.\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{array}\right\}$ $|J| = r$. A: $\begin{array}{l} 0 \le r \le 4 \\ 0 \le \varphi \le 2\pi \\ 0 \le z \le 4 - \underbrace{r\cos\varphi}_{x} \end{array}$

$$\iiint\limits_A \sqrt{x^2+y^2}\,dx\,dy = \int_0^4 \left(\int_0^{2\pi} \left(\int_0^{4-r\cos\varphi} r\cdot\sqrt{r^2\cos^2\varphi + r^2\sin^2\varphi}\,dz \right) d\varphi \right) dr =$$

$$= \int_0^4 \left(\int_0^{2\pi} r^2 \left[z \right]_0^{4-r\cos\varphi} d\varphi \right) dr = \int_0^4 \left(\int_0^{2\pi} r^2 (4 - r\cos\varphi)\,d\varphi \right) dr =$$

$$= \int_0^4 \left[4r^2\varphi - r^3\sin\varphi \right]_0^{2\pi} dr = \int_0^4 4r^2 2\pi\,dr = 8\pi \left[\frac{r^3}{3} \right] = \frac{512}{3}\pi$$

$\iiint\limits_A (1+xy)\,dx\,dy\,dz$, kde A: $x^2+y^2 \le z \le 2 - x^2 - y^2$



$x^2+y^2 = 2 - x^2 - y^2$

$2x^2 + 2y^2 = 2$

$x^2 + y^2 = 1$ — priemet do roviny x,y

$$= \int_0^1 \left(\int_0^{2\pi} \left(\int_{r^2}^{2-r^2} (1 + r^2\sin\varphi\cos\varphi)\cdot r\,dz \right) d\varphi \right) dr$$

$$= \int_0^1 \left(\int_0^{2\pi} \left[z \right]_{r^2}^{2-r^2} (r + r^3\sin\varphi\cos\varphi)\,d\varphi \right) dr$$

transf.: $\left.\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{array}\right\}$ $|J| = r$

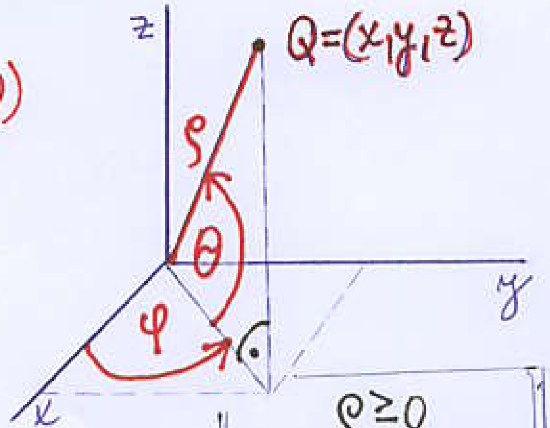
$$= \int_0^1 \left(\int_0^{2\pi} (2 - 2r^2)(r + r^3\sin\varphi\cos\varphi)\,d\varphi \right) dr$$

$$= \int_0^1 \left((2r - 2r^3)\left[\varphi \right]_0^{2\pi} + r^3 \underbrace{\left[\frac{\sin^2\varphi}{2} \right]_0^{2\pi}}_{0} \right) dr =$$

A: $\begin{array}{l} 0 \le r \le 1 \\ 0 \le \varphi \le 2\pi \\ r^2 \le z \le 2 - r^2 \end{array}$

$$= 2\pi \left[r^2 - 2\frac{r^4}{4} \right]_0^1 = 2\pi \left(1 - \frac{1}{2} \right) = \boxed{\pi}$$

Transformácia trojného integrálu do sférických súradníc



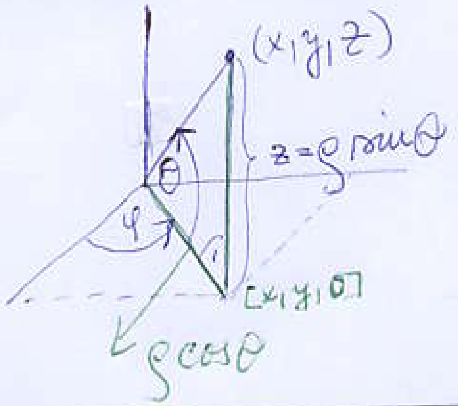
sférické súradnice: $(\varrho, \varphi, \theta)$

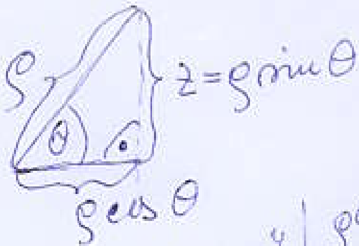
transf.:

$$\begin{cases} x = \varrho \cos\varphi \cos\theta \\ y = \varrho \sin\varphi \cos\theta \\ z = \varrho \sin\theta \end{cases}$$

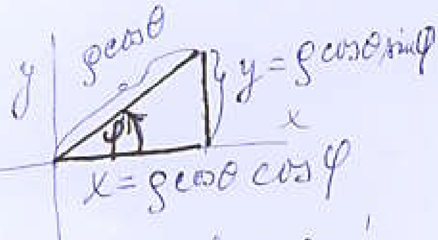
$$\boxed{\begin{gathered} \varrho \geq 0 \\ 0 \leq \varphi < 2\pi \\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \end{gathered}}$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial \varrho}, & \frac{\partial x}{\partial \varphi}, & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \varrho}, & \frac{\partial y}{\partial \varphi}, & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \varrho}, & \frac{\partial z}{\partial \varphi}, & \frac{\partial z}{\partial \theta} \end{vmatrix} = \varrho^2 \cdot \cos\theta$$





v rovine (x,y)



teda ak $r=\varrho\cos\theta$ tak r,φ sú polárne súradnice bodu (x,y) v rovine xy

Veta: Nech Ω: $\left.\begin{matrix} \alpha \leq \varphi \leq \beta \\ h_1(\varphi) \leq \theta \leq h_2(\varphi) \end{matrix}\right\} D$

$F_1(\varphi,\theta) \leq \varrho \leq F_2(\varphi,\theta)$

h_1, h_2 spojité funkcie na $\langle \alpha, \beta \rangle$

F_1, F_2 spojité funkcie na D

Nech $f \subseteq R^3 \times R$ je spojitá na Ω. Potom

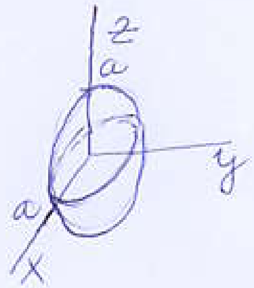
$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \int_\alpha^\beta \left(\int_{h_1(\varphi)}^{h_2(\varphi)} \left(\int_{F_1(\varphi,\theta)}^{F_2(\varphi,\theta)} f(\underbrace{\varrho\cos\varphi\cos\theta}_{x}, \underbrace{\varrho\sin\varphi\cos\theta}_{y}, \underbrace{\varrho\sin\theta}_{z}) \cdot \overbrace{\varrho^2\cos\theta}^{|J|}\, d\varrho \right) d\theta \right) d\varphi$$

pre typ $\langle \varphi, \theta, \varrho \rangle$

(ostatné typy Ω podobne.)

- Nech $G \subseteq R^3$ je guľa o polomere $\underline{a>0}$

$$G: \left.\begin{array}{r} 0 \le \varrho \le a \\ 0 \le \varphi \le 2\pi \\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \end{array}\right\}$$



Objem $C(G) = \iiint\limits_G dx\,dy\,dz =$

transf.: $x = \varrho\cos\theta\cos\varphi$
$y = \varrho\cos\theta\sin\varphi$
$z = \varrho\sin\theta$
$|J| = \varrho^2\cos\theta$

$$= \int_0^a \left(\int_0^{2\pi} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varrho^2 \cos\theta \, d\theta \right) d\varphi \right) d\varrho =$$

$$= \int_0^a \left(\int_0^{2\pi} \varrho^2 \left[\sin\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \right) d\varrho = \int_0^a \left(\int_0^{2\pi} 2\varrho^2 \, d\varphi \right) d\varrho =$$

$$= \int_0^a 2\varrho^2 \left[\varphi\right]_0^{2\pi} d\varrho = 4\pi \left[\frac{\varrho^3}{3}\right]_0^a = \frac{4\pi a^3}{3}$$

- Vypočítajme $\iiint\limits_G z^2 \, dx\,dy\,dz = \int_0^a \left(\int_0^{2\pi} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varrho^2\cos\theta \cdot \varrho^2 \sin^2\theta \, d\theta \right) d\varphi \right) d\varrho =$

$$= \int_0^a \left(\int_0^{2\pi} \varrho^4 \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \cos\theta \, d\theta \right) d\varphi \right) d\varrho =$$

$$= \int_0^a \left(\int_0^{2\pi} \varrho^4 \left[\frac{\sin^3\theta}{3}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \right) d\varrho = \int_0^a \left(\int_0^{2\pi} \varrho^4 \cdot \frac{2}{3} \, d\varphi \right) d\varrho =$$

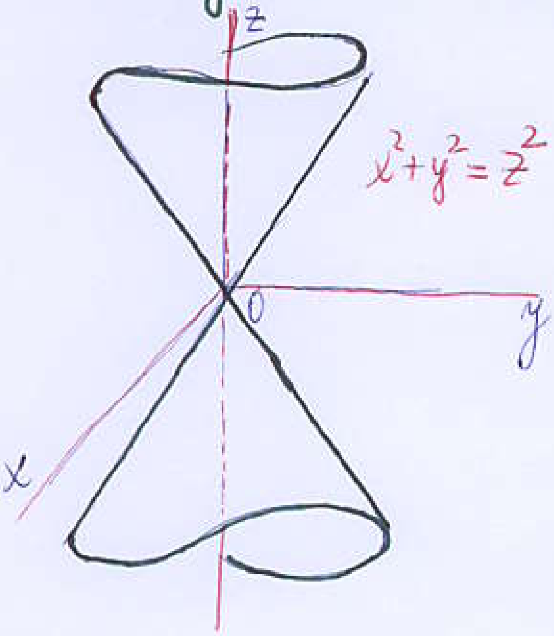
$$= \left[\frac{\varrho^5}{5}\right]_0^a \left[\frac{2}{3}\varphi\right]_0^{2\pi} = \frac{a^5}{5} \cdot \frac{2}{3} \cdot 2\pi = \frac{4a^5\pi}{15}$$

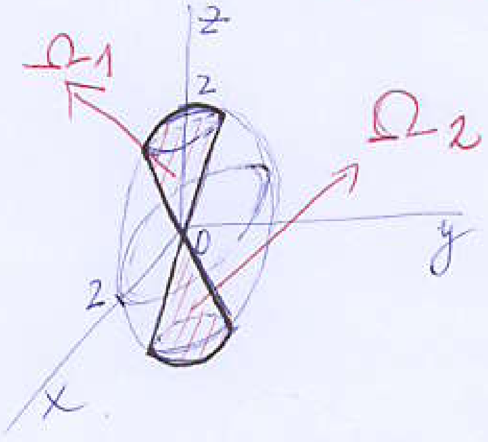
- $\iiint\limits_G y \, dx\,dy\,dz = \int_0^a \left(\int_0^{2\pi} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varrho^2\cos\theta \, \varrho\cos\theta \sin\varphi \, d\theta \right) d\varphi \right) d\varrho =$

$$= \left(\int_0^a \varrho^3 \, d\varrho \right) \underbrace{\left(\int_0^{2\pi} \sin\varphi \, d\varphi \right)}_{\left[-\cos\varphi\right]_0^{2\pi} = 0} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta \right) = \underline{0.}$$

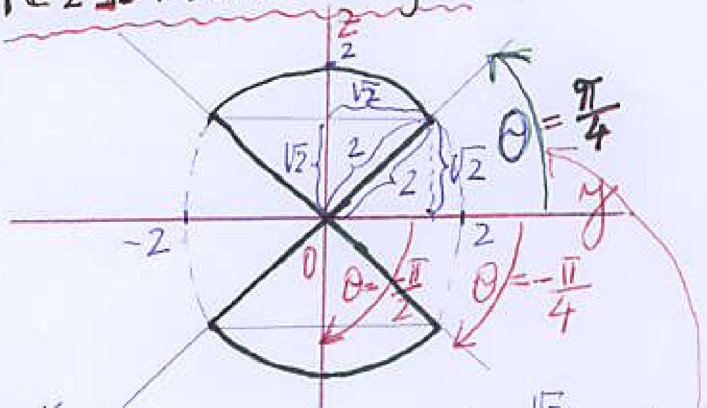
- Oblasť $\Omega \subseteq R^3$ je ohraničená guľovou plochou $x^2+y^2+z^2=4$ a dvojkužeľom $x^2+y^2=z^2$.

$\Omega = \Omega_1 \cup \Omega_2$





rez Ω rovinou y,z :



$$\begin{aligned} x^2+y^2+z^2 &= 4 \\ x^2+y^2 &= z^2 \\ \hline 2x^2+2y^2 &= 4 \\ x^2+y^2 &= 2 \end{aligned}$$

rez guľovej plochy a dvojkužeľa je kružnica o polomere $\sqrt{2}$

$\sin\theta = \frac{\sqrt{2}}{2} \Rightarrow$

$\Rightarrow \boxed{\theta = \frac{\pi}{4}}$

$$\Omega_1: \left.\begin{aligned} 0 &\le \rho \le 2 \\ 0 &\le \varphi \le 2\pi \\ \frac{\pi}{4} &\le \theta \le \frac{\pi}{2} \end{aligned}\right\}$$

nad rovinou x,y

$$\Omega_2: \begin{aligned} 0 &\le \rho \le 2 \\ 0 &\le \varphi \le 2\pi \\ -\frac{\pi}{2} &\le \theta \le -\frac{\pi}{4} \end{aligned}$$

pod rovinou x,y

$\underline{\Omega = \Omega_1 \cup \Omega_2}$

$\Omega_1 \cap \Omega_2 = \{0\}$

- $$C(\Omega) = 2C(\Omega_1) = 2\int_0^2 \left(\int_0^{2\pi} \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \rho^2 \cos\theta \, d\theta \right) d\varphi \right) d\rho =$$

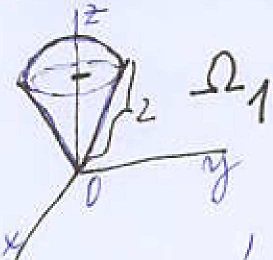
$|J|$

$$= 2\left(\int_0^2 \rho^2 d\rho\right)\left(\int_0^{2\pi} d\varphi\right)\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta \, d\theta\right) = 2\left[\frac{\rho^3}{3}\right]_0^2 \left[\varphi\right]_0^{2\pi} \left[\sin\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$$

$$= 2\cdot\frac{8}{3}\cdot 2\pi \left(1 - \frac{\sqrt{2}}{2}\right) = \boxed{\frac{16}{3}(2-\sqrt{2})\pi}$$

- $\displaystyle\iiint\limits_{\Omega_1} (x^2+y^2+z^2)^3\,dx\,dy\,dz$

kde Ω_1 je zo strany (24) ... časť oblasti Ω nad rovinou x,y.



transf.: $x = \varrho\cos\varphi\cos\theta$
$y = \varrho\sin\varphi\cos\theta$ $\quad |J| = \varrho^2\cos\theta$
$z = \varrho\sin\theta$

z transf. rovníc je:
$x^2+y^2 = \varrho^2\cos^2\theta(\cos^2\varphi+\sin^2\varphi) = \varrho^2\cos^2\theta$
$(x^2+y^2)+z^2 = \varrho^2\cos^2\theta+\varrho^2\sin^2\theta = \varrho^2$

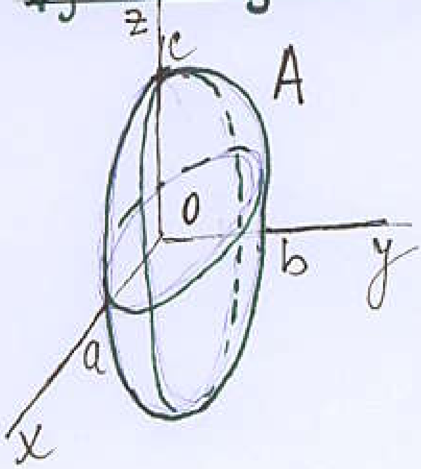
Ω_1: $0 \le \varrho \le 2$
$0 \le \varphi \le 2\pi$
$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$$\iiint\limits_{\Omega_1}(x^2+y^2+z^2)^3\,dx\,dy\,dz =$$
$$= \int_0^2\left(\int_0^{2\pi}\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\varrho^2)^3\cdot\varrho^2\cos\theta\,d\theta\right)d\varphi\right)d\varrho =$$
$$= \left(\int_0^2\varrho^6\cdot\varrho^2\,d\varrho\right)\left(\int_0^{2\pi}d\varphi\right)\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos\theta\,d\theta\right) =$$
$$= \left[\frac{\varrho^9}{9}\right]_0^2\left[\varphi\right]_0^{2\pi}\left[\sin\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$$
$$= \frac{2^9}{9}\cdot 2\pi\left(1-\frac{\sqrt{2}}{2}\right) = \boxed{\frac{2^9}{9}(2-\sqrt{2})\pi}$$

- $\displaystyle\iiint\limits_{\Omega}(x^2+y^2+z^2)^3\,dx\,dy\,dz = \iiint\limits_{\Omega_1}(x^2+y^2+z^2)^3\,dx\,dy\,dz +$
$\displaystyle + \iiint\limits_{\Omega_2}(x^2+y^2+z^2)^3\,dx\,dy\,dz = \boxed{\frac{2^{10}}{9}(2-\sqrt{2})\pi}$

vysvetlite!

Elipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $a>0, b>0, c>0$ a modifikácia transf. rovníc pomocou sférických súradníc



transf.:

$$\begin{cases} x = a\varrho\cos\varphi\cos\theta \\ y = b\varrho\sin\varphi\cos\theta \\ z = c\varrho\sin\theta \end{cases} \quad |J| = a.b.c.\varrho^2\cos\theta$$

$(\varrho>0,\ 0\le\varphi\le 2\pi,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2})$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = \varrho^2\cos^2\theta(\cos^2\varphi+\sin^2\varphi) + \varrho^2\sin^2\theta = \varrho^2 = 1$ na $A \Rightarrow$

$\Rightarrow 0\le\varrho\le 1$

Teda $\underline{\underline{A}}$: $\begin{cases} 0\le\varrho\le 1 \\ 0\le\varphi\le 2\pi \\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2} \end{cases}$

- $C(A) = \iiint\limits_A dx\,dy\,dz =$

$$= \int_0^1\left(\int_0^{2\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} abc\varrho^2\cos\theta\, d\theta\right)d\varphi\right)d\varrho =$$

$$= \left(\int_0^1 abc\varrho^2 d\varrho\right)\left(\int_0^{2\pi} d\varphi\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\, d\theta\right) =$$

$$= abc\left[\frac{\varrho^3}{3}\right]_0^1\left[\varphi\right]_0^{2\pi}\left[\sin\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= abc\cdot\frac{1}{3}\cdot 2\pi\,(1+1) = \boxed{\frac{4}{3}abc\pi}$$

- $\iiint\limits_A z\,dx\,dy\,dz = \int_0^1\left(\int_0^{2\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a.b.c\varrho^2\cos\theta\,.\,c\varrho\sin\theta\, d\theta\right)d\varphi\right)d\varrho =$

$$= \left(\int_0^1 abc^2\varrho^3 d\varrho\right)\left(\int_0^{2\pi} d\varphi\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\theta\cos\theta\, d\theta\right) = abc^2\left[\frac{\varrho^4}{4}\right]_0^1\cdot 2\pi\left[\frac{\sin^2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= abc^2\cdot\frac{1}{4}\cdot 2\pi\,\underbrace{\left(\frac{1}{2}-\frac{1}{2}\right)}_{0} = \underline{\underline{0}}$$