Vlastnosti bodov a podmnožín R^n

Zopakujme si:

Nech $\bar{a} = (a_1, a_2, \dots, a_n) \in R^n$, $A \subseteq R^n$, $\delta > 0$, potom:

$O_\delta(\bar{a}) = \{\bar{x} = (x_1, x_2, \dots, x_n) \in R^n \mid \|\bar{x} - \bar{a}\| < \delta\}$, okolie $\bar{a}$

$O^\circ_\delta(\bar{a}) = O_\delta(\bar{a}) \setminus \{\bar{a}\}$, prstencové okolie bodu $\bar{a}$

$\bar{a}$ je vnútorný bod A ak $\exists O_\delta(a) \subseteq A$

$\bar{a}$ je hraničný bod A ak $\forall \delta > 0$: $O_\delta(\bar{a}) \cap A \neq \emptyset$ a súčasne $O_\delta(\bar{a}) \cap (R^n \setminus A) \neq \emptyset$

$\bar{a}$ je hromadný bod A ak $\forall \delta > 0$: $O^\circ_\delta(\bar{a}) \cap A \neq \emptyset$

Množina A je:

otvorená ak každý jej bod je vnútorným bodom A

$Fr(A) = \{\bar{a} \in R^n \mid \bar{a}$ hraničný bod $A\}$ hranica A

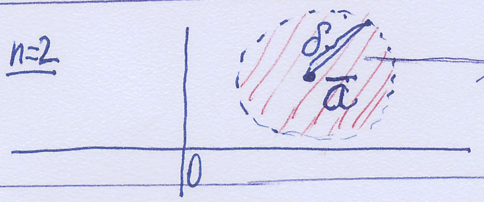
uzavretá ak $Fr(A) \subseteq A$

$\bar{A} = A \cup Fr(A)$

ohraničená ak existuje $r > 0$ že $A \subseteq O_r(\bar{0}) = \{\bar{x} \in R^n \mid \|\bar{x}\| < r\}$.

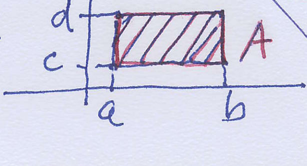
kompaktná ak A je ohraničená a uzavretá

$n=2$

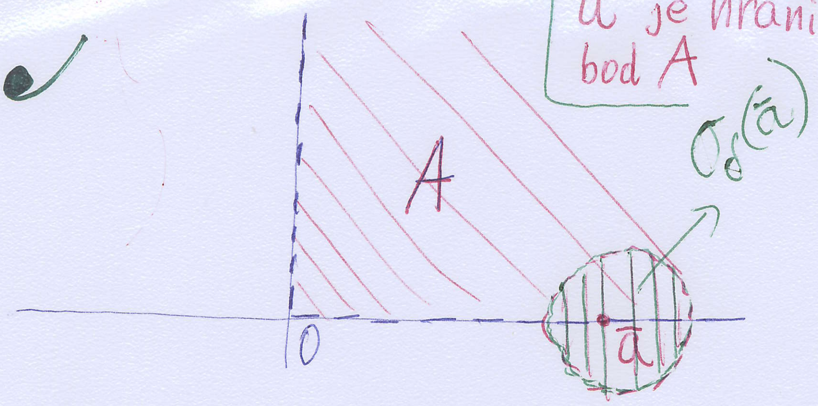


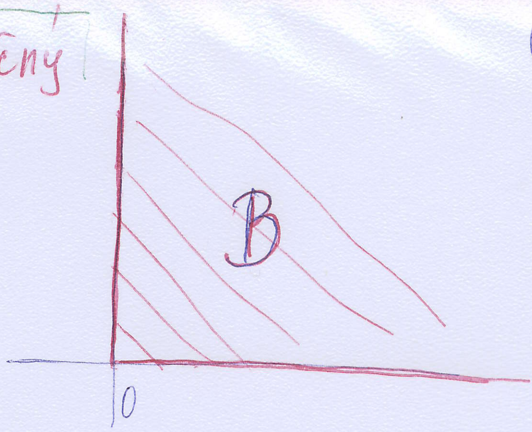
$O_\delta(\bar{a})$ je otvorená množina, je ohraničená, nie je kompaktná (nie je uzavretá)

$A = \langle a, b\rangle \times \langle c, d\rangle \subseteq R^2$ je kompaktná



$\bar{a}$ je hraničný bod A





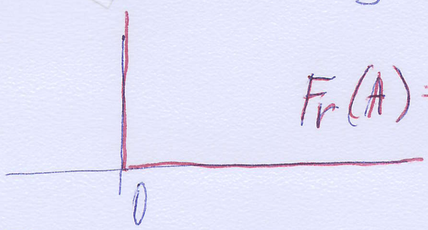
$A = \{(x,y) \in R^2 \mid x > 0, y > 0\}$, $B = \{(x,y) \in R^2 \mid x \geq 0, y \geq 0\}$

A — je otvorená; nie je ohraničená

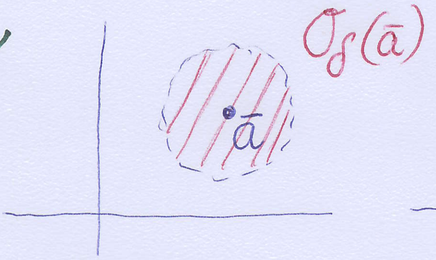
B — je uzavretá; nie je ohraničená $\Downarrow$ nie je kompaktná

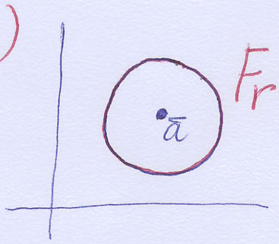
$Fr(A) = \langle 0, \infty) \times \{0\} \cup \{0\} \times \langle 0, \infty)$

$= \{(x,0) \in R^2 \mid x \geq 0\} \cup \{(0,y) \in R^2 \mid y \geq 0\} = Fr(B)$ — $\subseteq B$; $\not\subseteq A$

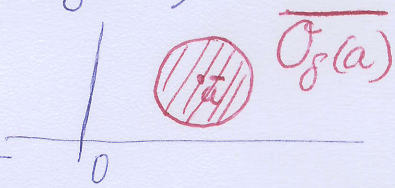


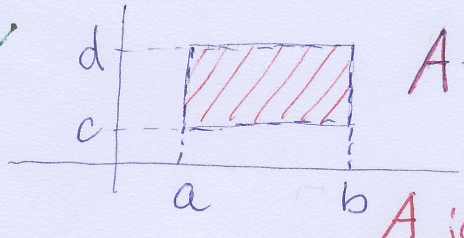
$Fr(A) = Fr(B)$ — $\subseteq B$; $\not\subseteq A$





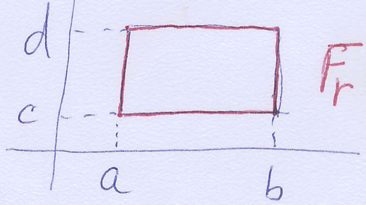
$Fr(O_\delta(\bar{a})) \not\subseteq O_\delta(\bar{a})$



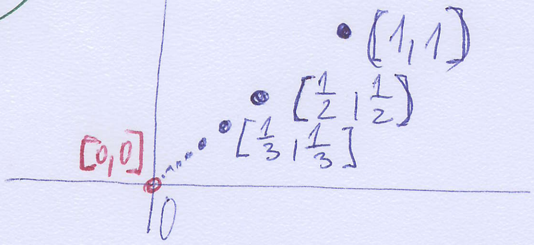


$A = (a,b) \times (c,d)$

A je otvorená



$Fr(A) \not\subseteq A$

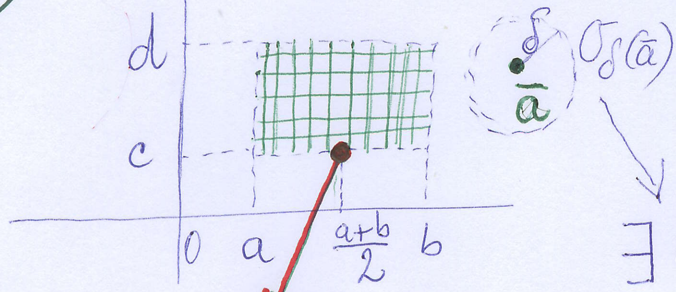


$A = \{[1,1], [\frac{1}{2}, \frac{1}{2}], [\frac{1}{3}, \frac{1}{3}], \dots\}$

bod $[0,0]$ je hromadný bod A

poznámka: $[0,0] \notin A$

$A = (a,b)\times(c,d) \cup \{\bar{a}\}$



$\bar{a}$ nie je hromadným bodom A

$\exists\, O^{\circ}_{\delta}(\bar{a}) : O^{\circ}_{\delta}(\bar{a}) \cap A = \emptyset$

bod $\left[\frac{a+b}{2}, c\right]$ je hromadným bodom A

zrejme: bod $\left[\frac{a+b}{2}, c\right]$ — nie je z A
— je z $Fr(A) \Rightarrow$
$\Rightarrow$ je z $\bar{A} = \langle a,b\rangle \times \langle c,d\rangle \cup \{\bar{a}\}$

bod $\bar{a}$ — je z $Fr(A) \Rightarrow$
$\Rightarrow$ je z $\bar{A}$
nie je hromadným bodom A (pretože $\exists O^{\circ}_{\delta}(\bar{a}) : O^{\circ}_{\delta}(\bar{a}) \cap A = \emptyset$)

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zopakujme si: Postupnosť bodov $(\bar{a}^{(k)})_{k=1}^{\infty}$, $\bar{a}^{(k)} = (a_1^{(k)}, a_2^{(k)}, \dots, a_n^{(k)}) \in R^n$ konverguje k bodu $\bar{a} = (a_1, a_2, \dots, a_n) \in R^n$ ak $\lim\limits_{k\to\infty} \|\bar{a}^{(k)} - \bar{a}\| = 0$

(označ: $\boxed{\lim\limits_{k\to\infty} \bar{a}^{(k)} = \bar{a}}$)

Platí: (1) $\lim\limits_{k\to\infty} \bar{a}^{(k)} = \bar{a} \iff \lim\limits_{k\to\infty} a_i^{(k)} = a_i$ pre $i = 1,2,3,\dots$

$\iff \forall \varepsilon > 0\ \exists k_0 \in N\ \forall k \geq k_0 : \|\bar{a}^{(k)} - \bar{a}\| < \varepsilon$

$\iff \forall O_\varepsilon(\bar{a})\ \exists k_0\ \forall k \geq k_0 : \bar{a}^{(k)} \in O_\varepsilon(\bar{a})$

(2) Bod $\bar{a} \in R^n$ je hromadným bodom množiny $A \subseteq R^n$ práve vtedy ak existuje postupnosť $(\bar{a}^{(k)})_{k=1}^{\infty}$, $\bar{a} \neq \bar{a}^{(k)}$, $\bar{a}^{(k)} \in A$ taká že $\lim\limits_{k \to \infty} \bar{a}^{(k)} = \bar{a}$.

- $\lim\limits_{k \to \infty} \left(\frac{1}{k}, \frac{1}{k}\right) = (0,0)$, pretože $\lim\limits_{k \to \infty} \frac{1}{k} = 0$

- $\lim\limits_{k \to \infty} \left(\frac{k}{k+1}, 1, \frac{k+1}{k}\right) = (1,1,1)$, pretože

$\lim\limits_{k \to \infty} \frac{k}{k+1} = 1$, $\lim\limits_{k \to \infty} 1 = 1$, $\lim\limits_{k \to \infty} \frac{k+1}{k} = 1$

bod $(1,1,1)$ je hrom. bodom tej postupnosti

- $\lim\limits_{k \to \infty} \left(\frac{k}{k+1}, 2k, \frac{(-1)^k}{k}\right)$ neexistuje

pretože $\lim\limits_{k \to \infty} 2k = \infty$

$\bar{A} = \left\{\left(\frac{k}{k+1}, 2k, \frac{(-1)^k}{k}\right) \middle| k = 1, 2 \dots\right\}$ nemá hromadné body

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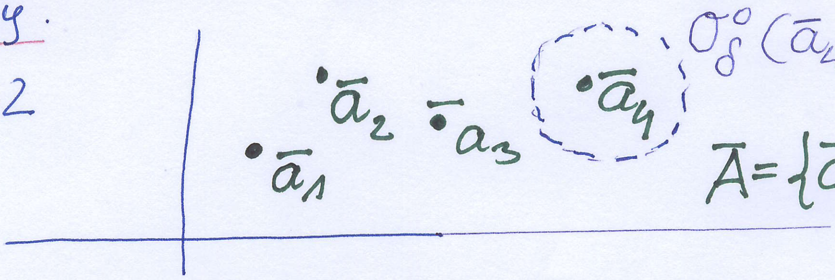
- Konečné podmnožiny R^n nemajú hromadné body.

napr. ak $n=2$



$O_\delta^0(\bar{a}_4) \cap A = \emptyset$

$\bar{A} = \{\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4\}$

pretože každý bod má prstencové okolie neobsahujúce body $\bar{A}$

Funkcia $f \subseteq R^n \times R$

$f \subseteq R^n \times R$ s vlastnosťou: $\boxed{(\bar{x}, y), (\bar{x}, z) \in f \Rightarrow y = z}$ *)

sa nazýva reálnou funkciou n-reálnych premenných.

*) (t.j. každému $\bar{x} \in R^n$ existuje najviac jedno $y \in R$ také že $(\bar{x}, y) \in f$ a potom píšeme $y = f(\bar{x})$ alebo $y = f(x_1, x_2, \dots, x_n)$).

$D(f) = \{\bar{x} \in R^n \mid$ existuje $y \in R$ že $(\bar{x}, y) \in f\} \subseteq R^n$

je obor definície f

$H(f) = \{y \in R \mid$ existuje $\bar{x} \in R^n$ že $(\bar{x}, y) \in f\} \subseteq R$

je množina hodnôt f

f je
- ohraničená ak $H(f) \subseteq R$ je ohraničená
- $\max f = \max H(f)$
- $\min f = \min H(f)$
- $\sup f = \sup H(f)$
- $\inf f = \inf H(f)$

ak existujú.

množinu bodov $f \subseteq R^{n+1} = R^n \times R$ nazývame grafom f

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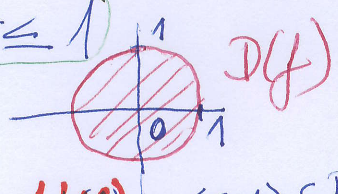
$f(x, y) = \sqrt{1 - x^2 - y^2}$

$(f \subseteq R^2 \times R)$

$\boxed{D(f) \subseteq R^2 : x^2 + y^2 \leq 1}$



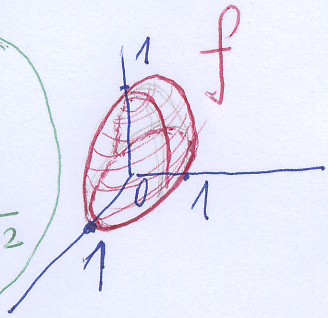
$(x, y) \in D(f) \Rightarrow$

$\Rightarrow 0 \leq \underbrace{f(x, y)}_{\sqrt{1 - x^2 - y^2}} \leq 1 \Rightarrow H(f) = \langle 0, 1 \rangle \subseteq R$

$[x, y, z] \in f$

$\Updownarrow$

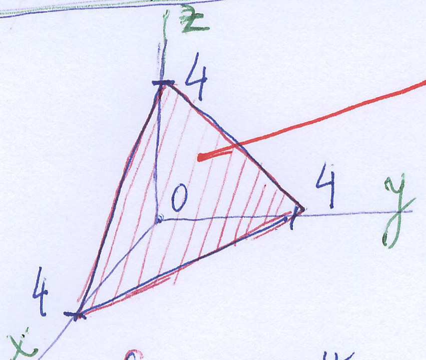
$z = \sqrt{1 - x^2 - y^2}$



grafom f je povrch pologule so stredom v začiatku a polomerom $r = 1$.

$\max f = 1, \min f = 0$

$$\boxed{f(x,y) = 4 - x - y\ ,\ x \geq 0, y \geq 0,\ z \geq 0}$$

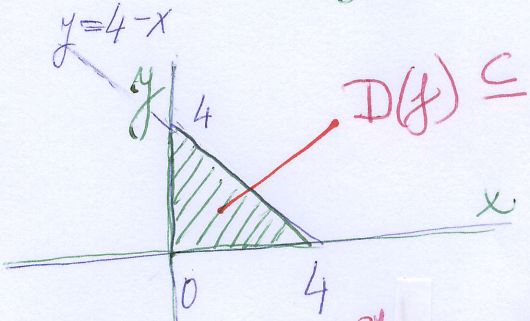


grafom f je množina

$[x,y,z] \in R^3 = R^2 \times R$

$z \overset{!}{=} \underbrace{4 - x - y}_{f(x,y)}$ & $x \geq 0, y \geq 0, z \geq 0$

grafom f je časť roviny $x+z+z=4$ v 1-vom oktante



$D(f) \subseteq R^2$

$H(f) = \langle 0, 4 \rangle$

$D(f) = \{[x,y] \in R^2 / 0 \leq x + y \leq 4\}$

napr.: $f(0,0) = 4 - 0 - 0 = 4 \in H(f)$

$f(4,0) = 4 - 4 - 0 = 0 \in H(f)$

$f(0,4) = 4 - 0 - 4 = 0 \in H(f)$

$f(1,1) = 4 - 1 - 1 = 2 \in H(f)$

poznámka: $H(f)$ tvoria z-tové súradnice bodov grafu f.

$\max f = 4$

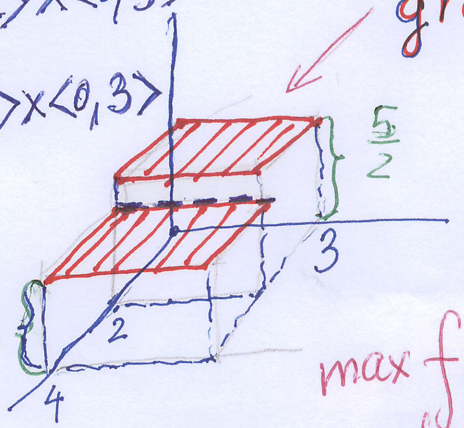
$\min f = 0$

$z = f(x,y)$, kde $x^2 + y^2 + z^2 = 4$ nie je funkcia

pretože pre $[x,y]$ pre ktoré $x^2 + y^2 \leq 4$ vyhovujú $z_1 = \sqrt{4 - x^2 - y^2}$, $z_2 = -\sqrt{4 - x^2 - y^2}$

$$f(x,y) = \begin{cases} \frac{5}{2} & \text{pre } [x,y] \in \langle 0,2\rangle \times \langle 0,3\rangle \\ 2 & \text{pre } [x,y] \in (2,4\rangle \times \langle 0,3\rangle \end{cases}$$

graf f



$D(f) = \langle 0,4 \rangle \times \langle 0,3 \rangle \subseteq R^2$

$H(f) = \{2, \frac{5}{2}\} \subseteq R$

$\max f = \frac{5}{2}$

$\min f = 2$

Limita a spojitosť funkcie $f \subseteq R^m \times R$ v bode $\bar{a}$ ktorého $\exists O_\delta^\circ(\bar{a}) \subseteq D(f)$ a na otvorenej množine $G \subseteq D(f)$

(1) $\lim\limits_{\bar{x} \to \bar{a}} f(\bar{x}) = b \in R \overset{\text{Def.}}{\Longleftrightarrow}$ pre každú postupnosť $(\bar{a}^{(k)})_{k=1}^{\infty}$, $\bar{a}^{(k)} \in D(f)$, $\bar{a}^{(k)} \neq \bar{a}$, $\bar{a}^{(k)} \xrightarrow[k \to \infty]{} \bar{a}$ platí: $\lim\limits_{k \to \infty} f(\bar{a}^k) = b$

(2) f je spojitá v $\bar{a} \overset{\text{Def.}}{\Longleftrightarrow} \lim\limits_{\bar{x} \to \bar{a}} f(\bar{x}) = f(\bar{a})$

(3) f je spojitá na $G \overset{\text{Def.}}{\Longleftrightarrow}$ f je spojitá v každom $\bar{a} \in G$

zostávajú v platnosti vety o $f+g$, $f \cdot g$, $\frac{f}{g}$, o súčine nulovej a ohraničenej funkcie

Platí: $\lim\limits_{\bar{x} \to \bar{a}} f(\bar{x}) = b \in R \Longleftrightarrow \forall O_\varepsilon(b)\ \exists O_\delta^\circ(\bar{a})\ \forall \bar{x} \in O_\delta^\circ(\bar{a}):$
$f(\bar{x}) \in O_\varepsilon(b)$

$\Longleftrightarrow \forall \varepsilon > 0\ \exists \delta > 0\ \forall\ 0 < \|\bar{x} - \bar{a}\| < \delta:$
$|f(\bar{x}) - b| < \varepsilon$

- $\lim\limits_{\substack{x \to 3 \\ y \to 2}} xy = \underline{\underline{6}}$, pretože ak $(x_k, y_k) \xrightarrow[k \to \infty]{} (3, 2)$
  teda $x_k \xrightarrow[k \to \infty]{} 3$, $y_k \xrightarrow[k \to \infty]{} 2$
  potom $\underbrace{x_k \cdot y_k}_{f(x_k, y_k) =} \xrightarrow[k \to \infty]{} 3 \cdot 2 = 6$

$$\lim_{(x,y)\to(2,4)} (x^2y - 3xy^3) = 2^2\cdot 4 - 3\cdot 2\cdot 4^3 = 16 - 384 = = \underline{-368}$$

pretože: ak $(x_k, y_k) \to (2,4)$ potom

$$\lim_{k\to\infty} \left((x_k)^2 \cdot y_k - 3x_k (y_k)^3\right) = 2^2\cdot 4 - 3\cdot 2\cdot 4^3 = = \underline{-368}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \underbrace{\frac{x}{e^x + y - 1}}_{= f(x,y)}$$ neexistuje, pretože:

$$\left\{\begin{array}{l} \left(0, \frac{1}{k}\right) \xrightarrow[k\to\infty]{} (0,0) \\ f\left(0, \frac{1}{k}\right) = \frac{0}{e^0 + \frac{1}{k} - 1} = 0 \xrightarrow[k\to\infty]{} 0 \end{array}\right.$$

$0 \neq 1 \implies$ daná limita neexistuje

$$\left\{\begin{array}{l} \left(\frac{1}{k}, 0\right) \xrightarrow[k\to\infty]{} (0,0) \\ f\left(\frac{1}{k}, 0\right) = \frac{\frac{1}{k}}{e^{\frac{1}{k}} + 0 - 1} = \frac{\frac{1}{k}}{e^{\frac{1}{k}}} \xrightarrow[k\to\infty]{} 1 \end{array}\right.$$

$$\left(\text{pretože } \underbrace{\frac{1}{k}}_{= x_k} \xrightarrow[k\to\infty]{} 0 \ \& \ \lim_{x\to 0} \frac{x}{e^x - 1} = \lim_{x\to 0} \frac{1}{e^x} = 1\right.$$

L'Hospitalovo pravidlo

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{\substack{x\to 0\\ y\to 0}} \left(\underbrace{x}_{\to 0} \cdot \underbrace{\frac{x^2}{x^2+y^2}}_{\leq 1} + \underbrace{y}_{\to 0} \cdot \underbrace{\frac{y^2}{x^2+y^2}}_{\leq 1}\right) = \underline{\underline{0}}$$

$\underline{\text{pretože}}$ $$\lim_{\substack{x\to 0\\ y\to 0}} x = 0 \ \& \left|\frac{x^2}{x^2+y^2}\right| \leq 1 \Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} x \cdot \frac{x^2}{x^2+y^2} = 0$$

a podobne $$\lim_{\substack{x\to 0\\ y\to 0}} y \cdot \frac{y^2}{x^2+y^2} = 0$$

(1) Limita funkcie $f \subseteq R^n \times R$

v hromadnom bode $\bar{a}$ množiny $A \subseteq D(f)$

cez množinu A

(označ.: $\lim\limits_{\bar{x} \to \bar{a}\,(\text{cez } A)} f(x) = b$)

(2) Spojitosť funkcie $f \subseteq R^n \times R$

v hromadnom bode $\bar{a}$ množiny $A \subseteq D(f)$

s ohľadom na A.

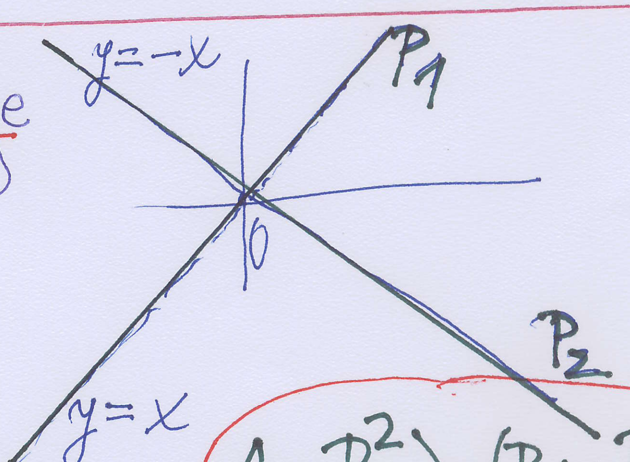
(1) $\lim\limits_{\bar{x} \to \bar{a}\,(\text{cez } A)} f(\bar{x}) = b \in R \overset{\text{Def.}}{\Longleftrightarrow} \forall\, \bar{a}^{(k)} \in A,\ \bar{a}^{(k)} \neq \bar{a},\ k = 1, 2, \dots$

$$\bar{a}^{(k)} \xrightarrow[k \to \infty]{} \bar{a} \Rightarrow f(\bar{a}^{(k)}) \xrightarrow[k \to \infty]{} b$$

(2) f je spojitá v hromadnom bode $\bar{a}$ množiny A s ohľadom na A (kde $A \subseteq D(f)$) ak

$$\lim_{\bar{x} \to \bar{a}\,(\text{cez } A)} f(\bar{x}) = f(\bar{a})$$

$\lim\limits_{\substack{(x,y) \to (0,0) \\ (\text{cez } A)}} \dfrac{x}{x^2 - y^2}$ neexistuje (vlastná)



pretože $\underbrace{\left(\frac{2}{k}, \frac{1}{k}\right)}_{\in A} \xrightarrow[k \to \infty]{} (0,0)$

$$f\left(\frac{2}{k}, \frac{1}{k}\right) = \frac{\frac{2}{k}}{\frac{4}{k^2} - \frac{1}{k^2}} = \frac{2}{3}k \xrightarrow[k \to \infty]{} \infty$$

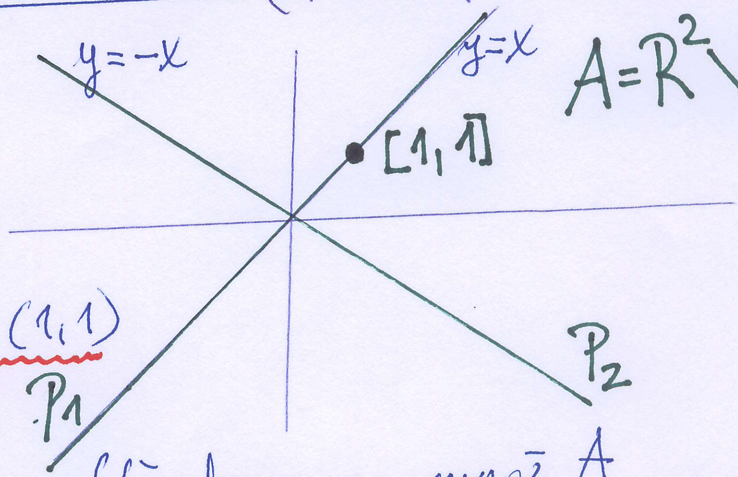
$A = R^2 \setminus (P_1 \cup P_2)$

- funkcia $f(x,y) = \dfrac{x}{x^2-y^2}$

sa nedá v bode $\bar{a} = [0,0]$ dodefinovať na "spojitú v bode $\bar{a}$ s ohľadom na množ. A"

pretože $\displaystyle\lim_{(x,y)\to(0,0)\,(\text{cez } A)} \frac{x}{x^2-y^2}$ neexistuje. (pozri predchádzajúci prík.)

$A = R^2 \setminus (P_1 \cup P_2)$



- $f(x) = \dfrac{x}{x^2-y^2}$

sa nedá v bode $\bar{a} = (1,1)$ dodefinovať na spojitú v $\bar{a} = (1,1)$ s ohľadom na množ. A

pretože

$$\lim_{\substack{(x,y)\to(1,1)\\(\text{cez } A)}} \frac{x}{x^2-y^2} \quad \text{neexistuje}$$

pretože: $\underbrace{\left(\dfrac{k+2}{k}, \dfrac{k+1}{k}\right)}_{\in A} \xrightarrow[k\to\infty]{} (1,1)$

$$f\left(\frac{k+2}{k}, \frac{k+1}{k}\right) = \frac{\frac{k+2}{k}}{\frac{(k+2)^2-(k+1)^2}{k^2}} = \frac{k^{\cancel{2}}(k+2)}{\cancel{k}(\cancel{k^2}+4k+4-\cancel{k^2}-2k-1)} =$$

$$= \frac{k^2+2k}{2k+3} \xrightarrow[k\to\infty]{} \infty$$

(podobne v ľubovoľnom bode $[a,b] \in P_1$).