Nevlastné integrály z neohraničenej funkcie

Zopakujme si: $\int_a^b f(x)\,dx$ nazývame určitým (vlastným) R-integrálom ak $a, b \in R$ a $|f(x)| \leq k$ na $\langle a, b \rangle$.

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(1) Ak
- $a, b \in R$
- $(a, b\rangle \subseteq D(f)$
- $\lim\limits_{x \to a+} f(x) = \infty \ (-\infty)$
- $\int_t^b f(x)\,dx$ existuje pre všetky $t \in (a, b\rangle$ (vlastný)

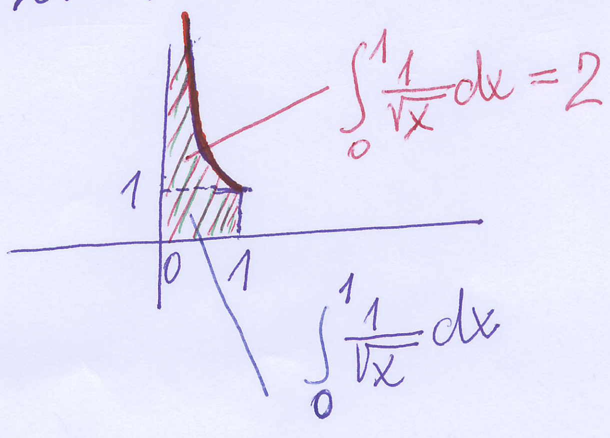
potom ak existuje $\boxed{\lim\limits_{t \to a+} \int_t^b f(x)\,dx \overset{\text{Def.}}{=} \int_a^b f(x)\,dx}$ (vlastná)

hovoríme že nevlastný integrál $\int_a^b f(x)\,dx$ konverguje (inak diverguje).

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2 \quad \text{pretože:}$$

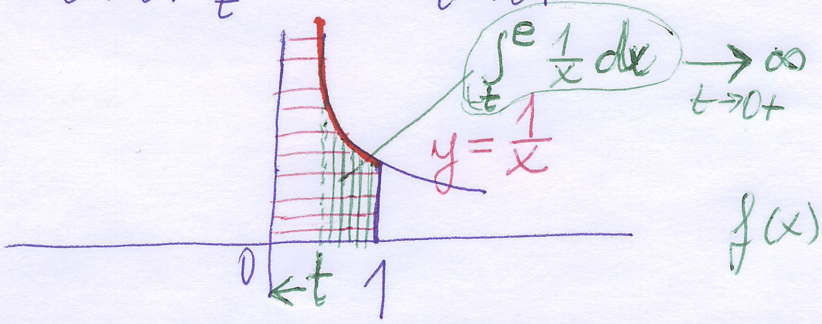
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t \to 0+} \int_t^1 x^{-\frac{1}{2}}\,dx = \lim_{t \to 0+} \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_t^1 =$$

$$= \lim_{t \to 0+} (2 - 2\sqrt{t}) = 2 \quad \text{(konverguje)}$$



- $\int_0^e \frac{1}{x}\,dx$ diverguje, pretože:

$$\lim_{t\to 0+}\int_t^e \frac{1}{x}\,dx = \lim_{t\to 0+}\left[\ln x\right]_t^e = \lim_{t\to 0+}(1-\ln t)=\infty$$



$f(x)=\frac{1}{x}$ je neohraničená na $O_\delta(0+)$

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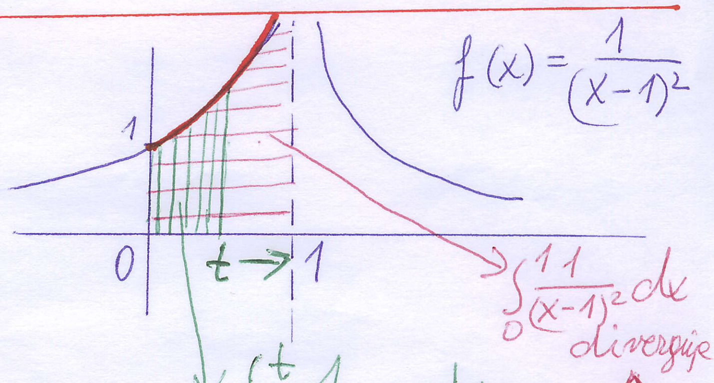
(2) Ak
- $a,b\in R$
- $\langle a,b)\subseteq D(f)$
- $\lim_{x\to b-} f(x)=\infty\ (-\infty)$
- $\int_a^t f(x)\,dx$ existuje pre všetky $t\in\langle a,b)$

potom ak existuje $\boxed{\lim_{t\to b-}\int_a^t f(x)\,dx \doteq \int_a^b f(x)\,dx}$ (vlastná)

hovoríme že nevlastný integrál konverguje (inak diverguje)

- $\int_0^1 \frac{1}{(x-1)^2}\,dx$ diverguje

$\frac{1}{(x-1)^2}$ neohraničená na $O_\delta(1-)$

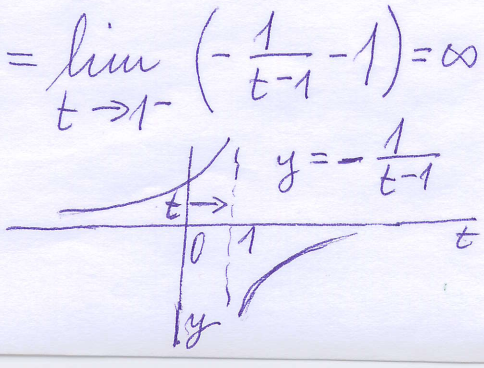


pretože:

$$\int_0^1 \frac{1}{(x-1)^2}\,dx = \lim_{t\to 1-}\int_0^\infty \frac{1}{(x-1)^2}\,dx =$$

$$= \lim_{t\to 1-}\left[-\frac{1}{x-1}\right]_0^t = \lim_{t\to 1-}\left(-\frac{1}{t-1}-1\right)=\infty$$

$$\int_0^t \frac{1}{(x-1)^2}\,dx = \left(-\frac{1}{t-1}-1\right)\xrightarrow[t\to 1-]{}\infty$$



Nevlastné integrály na neohraničenom intervale

(3) Ak $\begin{cases} a \in R, & \langle a, \infty) \subseteq D(f) \\ \int_a^t f(x)\,dx & \text{existuje pre všetky } \underline{t > a} \text{ (vlastný)} \end{cases}$

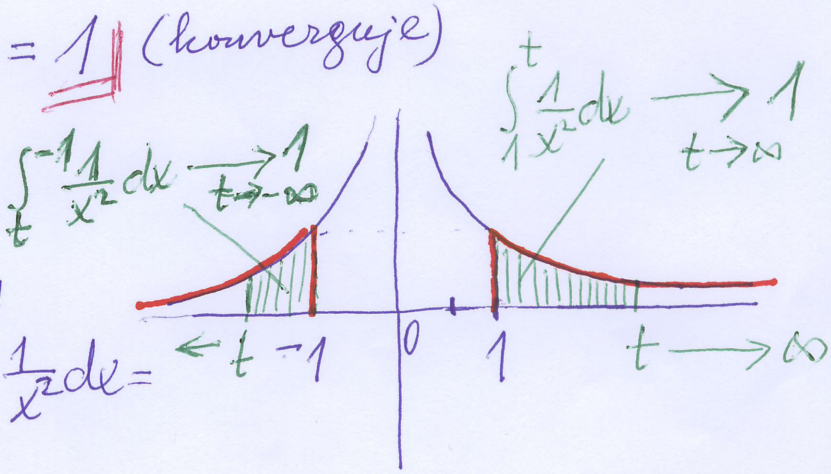
potom ak existuje (vlastná) $\lim\limits_{t\to\infty} \int_a^t f(x)\,dx \overset{\text{Def.}}{=} \int_a^\infty f(x)\,dx$ **konverguje**

(4) Ak $\begin{cases} a \in R, & (-\infty, a\rangle \subseteq D(f) \\ \int_t^a f(x)\,dx & \text{existuje pre všetky } \underline{t < a} \text{ (vlastný)} \end{cases}$

potom ak existuje (vlastná) $\lim\limits_{t\to-\infty} f(x)\,dx \overset{\text{Def.}}{=} \int_{-\infty}^a f(x)\,dx$ **konverguje**

- $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t =$$

$$= \lim_{t\to\infty} \left(-\frac{1}{t} + 1\right) = \underline{\underline{1}} \text{ (konverguje)}$$



- $$\int_{-\infty}^{-1} \frac{1}{x^2}\,dx = \lim_{t\to-\infty} \int_t^{-1} \frac{1}{x^2}\,dx =$$

$$= \lim_{t\to-\infty} \left[-\frac{1}{x}\right]_t^{-1} = \lim_{t\to-\infty} \left(1 - \frac{1}{t}\right) = \underline{\underline{1}}$$

Nevlastné integrály s niekoľkými "kritickými" bodmi

Ak $a, b \in R$, f je spojitá na (a,b)

$\lim\limits_{x\to a^+} f(x) = \infty(-\infty)$, $\lim\limits_{x\to b^-} f(x) = \infty(-\infty)$

(5) Potom ak pre nejaké $d \in (a,b)$ konvergujú oba integrály $\int_a^d f(x)dx$, $\int_d^b f(x)dx$ tak

$$\int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)dx$$ konverguje

(6) Ak funkcia f má na intervale cez ktorý integrujeme konečne veľa "kritických" bodov rozdelíme daný interval na zjednotenie intervalov, z korých každý má len jediný kritický bod a to ako koncový.

pripad 6

$$\int_{-\infty}^{\infty} \frac{1}{x}dx = \int_{-\infty}^{-e} \frac{1}{x}dx + \int_{-e}^{0} \frac{1}{x}dx + \int_{0}^{e} \frac{1}{x}dx + \int_{e}^{\infty} \frac{1}{x}dx$$

! práve ak všetky na pravej strane konvergujú

pretože, ako sme zistily (v predchadz. prikladoch) $\int_0^e \frac{1}{x}dx$ diverguje, tak $\int_{-\infty}^{\infty} \frac{1}{x}dx$ diverguje prípad (5)

$$\int_0^1 \frac{1-2x}{\sqrt{x-x^2}}\,dx \qquad = f(x)$$

$x-x^2=0$

$x(1-x)=0$

$\underbrace{x_1=0}, \underbrace{x_2=1}$ kritické body

$x-x^2>0$

$x(1-x)>0$

$$\left.\begin{array}{l} \left.\begin{array}{l} x>0 \\ 1-x>0 \end{array}\right\} \Rightarrow 0<x<1 \\ \left.\begin{array}{l} x<0 \\ 1-x<0 \end{array}\right\} \Rightarrow \emptyset \end{array}\right\} \Rightarrow D(f)=(0,1)$$

f je na $(0,1)$ spojitá

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zvoľme $d=\frac{1}{2}$, potom nevlastné integrály

$$\int_0^{\frac{1}{2}} \frac{1-2x}{\sqrt{x-x^2}}\,dx, \quad \int_{\frac{1}{2}}^1 \frac{1-2x}{\sqrt{x-x^2}}\,dx$$ majú práve jeden kritický bod

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$$\int \frac{1-2x}{\sqrt{x-x^2}}\,dx = \int \frac{2t\,dt}{t} = 2\int dt = 2t = 2\sqrt{x-x^2}+C$$

na intervale $(0,1)$

subst: $x-x^2=t^2$

$(1-2x)dx=2t\,dt$

$$\int_0^{\frac{1}{2}} \frac{1-2x}{\sqrt{x-x^2}}\,dx = \lim_{t\to 0+}\int_t^{\frac{1}{2}} \frac{1-2x}{\sqrt{1-x^2}}\,dx = \lim_{t\to 0+}\left[2\sqrt{x-x^2}\right]_t^{\frac{1}{2}} =$$

pretože $\langle t, \frac{1}{2}\rangle \subseteq (0,1)$

$$= \lim_{t\to 0+}\left(\frac{2}{2}-2\sqrt{t-t^2}\right) = \underline{\underline{1}}$$

pretože $\langle \frac{1}{2}, t\rangle \subseteq (0,1)$

$$\int_{1/2}^1 \frac{1-2x}{\sqrt{x-x^2}}\,dx = \lim_{t\to 1-}\int_{1/2}^t \frac{1-2x}{\sqrt{1-x^2}}\,dx = \lim_{t\to 1-}\left[2\sqrt{x-x^2}\right]_{1/2}^t =$$

$$= \lim_{t\to 1-}\left(2\sqrt{t-t^2}-1\right) = \underline{\underline{-1}}$$

Teda $\boxed{\int_0^1 \frac{1-2x}{\sqrt{x-x^2}}\,dx = 1-1=0}$

$$\int_0^2 \frac{1}{(x-1)^2}\,dx \quad \text{diverguje}$$

pretože $\displaystyle\int_0^{1} \frac{1}{(x-1)^2}$ diverguje

f má na $\langle 0, 2\rangle$ kritický bod $c=1$

$$\int_{-1}^{1} \frac{1}{\sqrt{|x|}}\,dx = \int_{-1}^{0} \frac{1}{\sqrt{|x|}}\,dx + \int_{0}^{1} \frac{1}{\sqrt{|x|}}\,dx = 2+2 = \underline{4}$$

vypočítali sme že $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$

$x = |x|$ na $\langle 0,1\rangle$

$$\int_{-1}^{0} \frac{1}{\sqrt{|x|}}\,dx = \int_{-1}^{0} \frac{1}{\sqrt{(-x)}}\,dx = \lim_{t\to 0-} \int_{-1}^{t} \frac{1}{\sqrt{-x}}\,dx = \underline{\underline{2}}$$

subst: $u = -x$
$du = -dx$
$x = -1 \Rightarrow u = 1$
$x = t \Rightarrow u = -t$

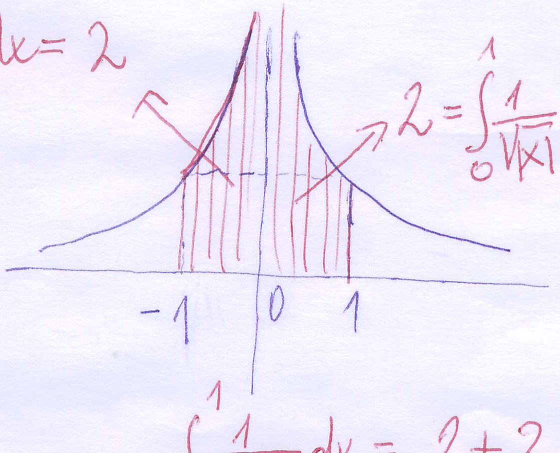
$$\int_{-1}^{t} \frac{1}{\sqrt{-x}}\,dx = \int_{1}^{-t} \frac{1}{\sqrt{u}}(-1)\,du =$$

$$= \left[-2\sqrt{u}\right]_1^{-t} = -2\sqrt{-t} + 2$$

$$\lim_{t\to 0-} \int_{-1}^{t} \frac{1}{\sqrt{-x}}\,dx = \lim_{t\to 0-} \left(-2\sqrt{-t} + 2\right) =$$

$$= 0 + 2 = \underline{\underline{2}}$$

$$\int_{-1}^{0} \frac{1}{\sqrt{x}}\,dx = 2$$



$$2 = \int_0^1 \frac{1}{\sqrt{|x|}}\,dx$$

$$\int_{-1}^{1} \frac{1}{\sqrt{x}}\,dx = 2+2 = \underline{\underline{4}}$$