Substitučná metóda

Zopakujme si:

(1) Substitúcia $t=\varphi(x)$ do $\int f(\varphi(x)).\varphi'(x)dx$:

Nech – F je primitívna k f na (α,β)
– φ má deriváciu φ' na (a,b)
– pre každé $x\in(a,b)$ je $\varphi(x)\in(\alpha,\beta)$

potom $\int f(\varphi(x)).\varphi'(x)\,dx = \int f(t)\,dt\Big|_{t=\varphi(x)} = F(\varphi(x))+C$ na (a,b)

pretože:
$\left([F(\varphi(x))]' = f(\varphi(x)).\varphi'(x)\right.$ pre každé $\left.x\in(a,b)\right)$

$$\int \frac{1}{x\ln x}\,dx = \int \underbrace{\frac{1}{\ln x}}_{f(\varphi(x))}\cdot\underbrace{\frac{1}{x}}_{\varphi'(x)}\,dx = \int \underbrace{\frac{1}{t}}_{f(t)}\,dt = \ln|t|\Big|_{t=\ln x} = \ln|\ln x|+C$$

$t=\ln x$
$dt=\frac{1}{x}dx$

na intervaloch $(0,1)$, $(1,\infty)$

napr.
$$\int_{\frac12}^{\frac34}\frac{1\,dx}{x\ln x} = \Big[\ln|\ln x|\Big]_{\frac12}^{\frac34} = \ln\left(\left|\ln\tfrac34\right|\right)-\ln\left(\left|\ln\tfrac12\right|\right) = \ln\left(\ln\tfrac43\right)-\ln(\ln 2)$$

(pretože: $\left|\ln\frac34\right| = |\ln 3-\ln 4| = \ln 4-\ln 3 = \ln\frac43$
a podobne $\left|\ln\frac12\right| = |\underbrace{\ln 1}_{0}-\underbrace{\ln 2}_{>0}| = \ln 2$) pretože $\langle\frac12,\frac34\rangle\subseteq(0,1)$

napr.
$$\int_2^e\frac{1}{x\ln x}\,dx = \Big[\ln|\ln x|\Big]_2^e = \ln|\underbrace{\ln e}_{1}|-\ln|\ln 2| = \underbrace{\ln 1}_{0}-\ln(\ln 2) = -\ln(\ln 2)$$

pretože $\langle 2,e\rangle\subseteq(1,\infty)$

$\cancel{\int_{\frac12}^{\frac32}\frac{1}{x\ln x}dx}$ nemôžeme uvažovať pretože $\langle\frac12,\frac32\rangle\not\subseteq(0,1)$ ani $(1,\infty)$

$$\int \frac{2x}{\sqrt{3x^2+9}}\,dx = \frac{1}{3}\int \frac{6x}{\sqrt{3x^2+9}}\,dx = \frac{1}{3}\int \frac{2t}{\sqrt{t^2}}\,dt = \frac{1}{3}\int 2\,dt =$$

$t^2 = 3x^2+9 \Rightarrow t = \sqrt{3x^2+9}$

$2t\,dt = 6x\,dx$

$$= \frac{2}{3}t + C = \frac{2}{3}\sqrt{3x^2+9} + C$$

na intervale $(-\infty, \infty)$

(skúška: $\left[\frac{2}{3}\sqrt{3x^2+9}\right]' = \frac{2}{3}\frac{1}{2\sqrt{3x^2+9}}\cdot 6x = \frac{2x}{\sqrt{3x^2+9}}$ na intervale $(-\infty, \infty)$)

$$\int (3x+1)\sqrt{3x-1}\,dx = \int (t+2)\sqrt{t}\cdot\frac{1}{3}\,dt = \frac{1}{3}\int \left(t^{\frac{3}{2}} + 2t^{\frac{1}{2}}\right)dt =$$

$\left(\begin{array}{l} t = 3x-1 \Rightarrow 3x+1 = t+2 \\ dt = 3\,dx \Rightarrow \frac{1}{3}\,dt = dx \end{array}\right)$

$$= \frac{1}{3}\left(\frac{2t^{\frac{5}{2}}}{5} + 2\frac{2t^{\frac{3}{2}}}{3}\right) + C = \frac{1}{3}\left(\frac{2}{5}\sqrt{(3x-1)^5} + \frac{4}{3}\sqrt{(3x-1)^3}\right) + C$$

na intervale $\left(\frac{1}{3}, \infty\right)$

napr. $$\int_1^3 (3x+1)\sqrt{3x-1}\,dx = \left[\frac{1}{3}\left(\frac{2}{5}\sqrt{(3x-1)^5} + \frac{4}{3}\sqrt{(3x-1)^3}\right)\right]_1^3 =$$

$$= \underbrace{\frac{1}{3}\left(\frac{2}{5}\cdot 8^{\frac{5}{2}} + \frac{4}{3}\cdot 8^{\frac{3}{2}}\right)}, \quad \text{pretože } \langle 1,3\rangle \subseteq \left(\frac{1}{3}, \infty\right)$$

napr. $$\cancel{\int_{-1}^0 (3x+1)\sqrt{3x-1}\,dx}$$ nemôžeme uvažovať pretože $\langle -1, 0\rangle \not\subseteq \left(\frac{1}{3}, \infty\right)$

$\sqrt{3x-1}$ je definované pre $3x-1 > 0$

$x > \frac{1}{3}$

$$\int tg\,x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{(-\sin x)}{\cos x}\,dx = -\ln|\cos x| + C$$

subst.: $t = \cos x$ na intervaloch $\left(-\frac{\pi}{2} + k\cdot\pi, \frac{\pi}{2} + k\cdot\pi\right)$, $k = 0, \pm 1, \pm 2, \dots$

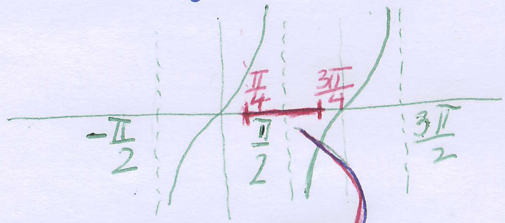
teda $\dots \left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right), \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right), \dots$

napr.:
$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} tg\,x\,dx = \Big[\ln|\cos x|\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \ln\left(\cos\frac{\pi}{4}\right) - \ln\left(\cos\left(-\frac{\pi}{4}\right)\right) = \ln\frac{\sqrt{2}}{2} - \ln\frac{\sqrt{2}}{2} = 0$$

$$\int_{0}^{\frac{\pi}{4}} tg\,x\,dx = \Big[\ln \cos x\Big]_{0}^{\frac{\pi}{4}} = \ln\left(\cos\frac{\pi}{4}\right) - \ln(\cos 0) = \ln\frac{\sqrt{2}}{2} - \underbrace{\ln 1}_{0} = \ln\frac{\sqrt{2}}{2}$$

pretože $\left\langle -\frac{\pi}{4}, \frac{\pi}{4} \right\rangle \subseteq \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\left\langle 0, \frac{\pi}{4} \right\rangle \subseteq \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



určitý R-int.: $\cancel{\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} tg\,x\,dx}$ nemôžeme uvažovať

pretože $\left\langle \frac{\pi}{4}, \frac{3\pi}{4} \right\rangle \not\subseteq \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$ pre žiadne celé k

$f(x) = tg\,x$ je tam neohraničená!

$$\int \cot g\,x\,dx = \int \frac{\cos x}{\sin x}\,dx = \cdots$$ riešime podobne

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$$\int \sin^4 x \cos x\,dx = \int t^4\,dt = \frac{t^5}{5}\Big|_{t=\sin x} = \frac{\sin^5 x}{5} + C$$

subst.: $t = \sin x$
$dt = \cos x\,dx$

na intervale $(-\infty, \infty)$

$$\int \sin^3 x \cos^4 x\,dx = \int \sin^2 x \cos^4 x \sin x\,dx =$$

subst.: $t = \cos x$
$dt = -\sin x\,dx$
$(-1)dt = \sin x\,dx$

$$= \int (1 - \cos^2 x)\cos^4 x \sin x\,dx =$$
$$= \int (1 - t^2)\,t^4\,(-1)\,dt =$$
$$= \int (-t^4 + t^6)\,dt = -\frac{t^5}{5} + \frac{t^7}{7} + C =$$
$$= -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$$ na $(-\infty, \infty)$

substitúcia $x = \varphi(t)$ do $\int f(x)\,dx$

zopakujme si:

Nech
- f je spojitá na (a,b)
- pre všetky $t \in (\alpha,\beta)$ je $\varphi(t) \in (a,b)$ a existuje $\varphi'(t) \neq 0$ pre všetky $t \in (\alpha,\beta)$, spojitá φ' na (α,β)
- nech $t = \varphi^{-1}(x)$ je inverzná k φ na (a,b)

potom

$$\int f(x)\,dx = \int f(\varphi(t))\varphi'(t)\,dt \Big|_{t=\varphi^{-1}(x)}$$

- $a>0$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\,dt}{\sqrt{a^2-a^2t^2}} = \int \frac{\cancel{a}}{\cancel{a}\sqrt{1-t^2}}\,dt = \int \frac{1}{\sqrt{1-t^2}}\,dt =$$

substit.: $x = at \Rightarrow t = \frac{x}{a}$, $dx = a\,dt$

$$= \arcsin t \Big|_{t=\frac{x}{a}} + C = \arcsin\frac{x}{a} + C$$

na $(-a,a)$

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$$\int 3\sqrt{1-x^2}\,dx = \int 3\sqrt{1-\sin^2 t}\,\cos t\,dt = 3\int \cos^2 t\,dt$$

subst.: $x = \sin t$, $dx = \cos t\,dt$ $\Rightarrow t = \arcsin x$ na $(-1,1)$

$$= 3\int \frac{1+\cos 2t}{2}\,dt = \frac{3}{2}\int (1+\cos 2t)\,dt =$$

$$= \frac{3}{2}\left(t + \frac{\sin 2t}{2}\right)\Big|_{t=\arcsin x} =$$

$$= \frac{3}{2}\left(\arcsin x + \frac{1}{\cancel{2}}\cancel{2}\,x\sqrt{1-x^2}\right)$$

$$= \frac{3}{2}\left(\arcsin x + x\sqrt{1-x^2}\right)$$

na intervale $(-1,1)$

pretože $\sin 2t = 2\sin t\cos t \Big|_{t=\arcsin x}$

$$= 2\sin(\arcsin x)\sqrt{1-\sin^2(\arcsin x)}$$

$$= 2x\sqrt{1-x^2}$$

Substitučná metóda pre určité integrály

Zopakujme si: substitúcia $t = \varphi(x)$

Nech
- f je spojitá na $\langle \alpha, \beta \rangle$
- φ má spojitú deriváciu $\varphi' \neq 0$ na $\langle a, b \rangle$
- pre každé $x \in \langle a, b \rangle$ je $\varphi(x) \in \langle \alpha, \beta \rangle$

pričom $\varphi(a) = \alpha$, $\varphi(b) = \beta$

potom $$\int_a^b f(\varphi(x))\,\varphi'(x)\,dx = \int_{\alpha=\varphi(a)}^{\beta=\varphi(b)} f(t)\,dt$$

- $$\int_1^e \frac{\ln^3 x}{x}\,dx = \int_1^e (\ln^3 x)\cdot\frac{1}{x}\,dx = \int_{0=\ln 1}^{1=\ln e} t^3\,dt = \left[\frac{t^4}{4}\right]_0^1 = \frac{1}{4}$$

subst.: $t = \ln x \Rightarrow \ln^3 x = t^3$

$dt = \frac{1}{x}\,dx$

ak $x = 1$ tak $t = \ln 1 = 0$

ak $x = e$ tak $t = \ln e = 1$

$x \in \langle 1, e \rangle \iff \ln x \in \langle 0, 1 \rangle$

- : $$\int_0^1 \frac{e^{2x}}{\sqrt[4]{e^x+1}}\,dx = \int_0^1 \frac{e^x}{\sqrt[4]{e^x+1}}\,e^x\,dx = \int_{\sqrt[4]{2}=\sqrt[4]{e^0+1}}^{\sqrt[4]{e+1}=\sqrt[4]{e^1+1}} \frac{t^4-1}{t}\,4t^3\,dt =$$

$t = \sqrt[4]{e^x+1}$

$t^4 = e^x + 1 \Rightarrow e^x = t^4 - 1$

$4t^3\,dt = e^x\,dx$

$$= \int_{\sqrt[4]{2}}^{\sqrt[4]{e+1}} \left(t^3 - \frac{1}{t}\right) 4t^3\,dt$$

$$= 4\left[t^6 - t^2\right]_{\sqrt[4]{2}}^{\sqrt[4]{e+1}} =$$

$$= 4\{\underbrace{(e+1)\sqrt{e+1}}_{(e+1)^{\frac{6}{4}} = (e+1)^{\frac{3}{2}} = (e+1)\sqrt{e+1}} - \sqrt{e+1} - 2\sqrt{2} + \sqrt{2}\} = 4(e\sqrt{e} - \sqrt{2})$$

$$\int\sqrt{5-4x-x^2}\,dx = \int\sqrt{9-(x+2)^2}\,dx =$$

<u>subst:</u> $x+2 = 3\sin t \Rightarrow t = \arcsin\frac{x+2}{3}$

$dx = 3\cos t\,dt$

$$= \int\underbrace{\sqrt{9-9\sin^2 t}}_{3\cos t}\cdot 3\cos t\,dt = \int 9\cdot\cos^2 t\,dt =$$

$$= 9\int\frac{1+\cos 2t}{2}\,dt = \frac{9}{2}\left(t+\frac{\sin 2t}{2}\right)\Big/_{t=\arcsin\frac{x+2}{3}} =$$

$$\frac{9}{2}\left(\arcsin\frac{x+2}{3} + \frac{x+2}{3}\sqrt{1-\left(\frac{x+2}{3}\right)^2}\right)+C$$

$$\frac{\sin 2t}{2} = \frac{2\sin t\cos t}{2} = \sin t\cos t =$$

$$= (\sin t)\sqrt{1-\sin^2 t}$$

<u>na intervale:</u> $(-5, 1)$

$$9-(x+2)^2>0$$
$$(x+2)^2<9$$
$$|x+2|<3$$
$$-3<x+2<3$$
$$-5<x<1$$

$$\int_{-\pi}^{0}\sqrt{5-4x-x^2}\,dx = \int_{-\pi}^{0}\sqrt{9-(x+2)^2}\,dx = \int_{\arcsin\frac{-\pi+2}{3}}^{\arcsin\frac{2}{3}} 9\cos^2 t\,dt =$$

<u>subst.:</u> $x+2 = 3\sin t$

$dx = 3\cos t$

$$= \left[\frac{9}{2}\left(t+\frac{\sin 2t}{2}\right)\right]_{\arcsin\frac{-\pi+2}{3}}^{\arcsin\frac{2}{3}}$$

$$= \frac{9}{2}\left(\arcsin\frac{2}{3} - \arcsin\frac{-\pi+2}{3} + \frac{2}{3}\sqrt{1-\left(\frac{2}{3}\right)^2} - \left(-\frac{\pi+2}{3}\right)\cdot\sqrt{1-\left(-\frac{\pi+2}{3}\right)^2}\right)$$

$$\int \frac{1}{2\sqrt{x}} \cos\sqrt{x}\, dx = \int \cos t\, dt = \sin t + C \Big/_{t=\sqrt{x}} =$$

subst.: $t = \sqrt{x}$

$dt = \frac{1}{2\sqrt{x}} dx$

$$= \underline{\sin\sqrt{x} + C}$$

na intervale $(0,\infty)$

$$\int_{\pi^2}^{\pi^4} \frac{1}{2\sqrt{x}} \cos\sqrt{x}\, dx = \int_{\pi=\sqrt{\pi^2}}^{\pi^2=\sqrt{\pi^4}} \cos t\, dt = \Big[\sin t\Big]_{\pi}^{\pi^2} =$$

subst.: $t = \sqrt{x}$

$$= \underline{\sin\pi^2 - \sin\pi}$$

$$\int \cos\sqrt{x}\, dx = \int \underbrace{(\cos t)}_{f'} . \underbrace{2t}_{g'}\, dt = \underbrace{(\sin t)}_{f} . \underbrace{2t}_{g} - \int \underbrace{(\sin t)}_{f} \underbrace{2}_{g'}\, dt =$$

subst.: $x = t^2$

$dx = 2t\,dt$

$$= 2t \sin t + 2\cos t \Big/_{t=\sqrt{x}} = \underline{2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C}$$

na intervale $(0,\infty)$

$$\int \frac{x}{(x^2+1)^2} dx = \frac{1}{2}\int \frac{2x\, dx}{(x^2+1)^2} = \frac{1}{2}\int \frac{dt}{t^2} = \frac{1}{2}\frac{t^{-1}}{-1} =$$

subst.: $x^2+1 = t$

$2x\, dx = dt$

$$= -\frac{1}{2t}\Big/_{t=x^2+1} = -\frac{1}{2(x^2+1)} + C$$

na intervale $(-\infty,\infty)$

$$\int_0^1 \frac{x}{(x^2+1)^2} dx = \frac{1}{2}\int_1^2 \frac{dt}{t^2} = \Big[-\frac{1}{2t}\Big]_1^2 = -\frac{1}{4} + \frac{1}{2} = \underline{\frac{1}{4}}$$

subst.: $x^2+1 = t$

$2x\, dx = dt$