1

6.11.

$$\boxed{\begin{aligned} tg\frac{x}{2} &= t \\ \sin x &= \frac{2t}{1+t^2} \\ \cos x &= \frac{1-t^2}{1+t^2} \\ dx &= \frac{2}{1+t^2}\,dt \end{aligned}}$$

$$\int \frac{1}{(2+\cos x)\sin x}\,dx = \int \frac{1}{\left(2+\frac{1-t^2}{1+t^2}\right)\frac{2t}{1+t^2}}\cdot\frac{2}{(1+t^2)}\,dt = \cancel{2}\int \frac{1}{\frac{2+2x^2+1-x^2}{1+x^2}\cdot\frac{\cancel{2}t}{\cancel{1+t^2}}}\cdot\frac{1}{(1+t^2)}\,dt =$$

$$= \int \frac{t^2+1}{(t^2+3)t}\,dt$$

$$\frac{t^2+1}{(t^2+3)t} = \frac{At+B}{t^2+3} + \frac{C}{t}$$

$$\int \frac{dx}{3+\cos x} = \int \frac{1}{3+\frac{1-t^2}{1+t^2}}\cdot\frac{2dt}{1+t^2} = \int \frac{1}{\frac{3+3t^2+1-t^2}{1+t^2}}\cdot\frac{2}{\cancel{1+t^2}}\,dt = 2\int \frac{1}{2t^2+4}\,dt = \frac{1}{2}\int \frac{1}{2(t^2+2)}\cdot dt =$$

$$= \int \frac{1}{\left(\frac{t}{\sqrt{2}}\right)^2+1}\,dt = \int \frac{1}{1+z^2}\,dz\,\sqrt{2} =$$

$$\frac{t}{\sqrt{2}} = z \qquad dt = dz\cdot\sqrt{2}$$

$$\int \frac{\sin x-\cos x}{\sin x+2\cos x}\,dx = \int \frac{\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}+\frac{2(1-t^2)}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int \frac{\frac{t^2+2t-1}{1+t^2}}{\frac{2t+2(-t^2+t+1)}{1+t^2}}\,\frac{2}{1+t^2}\,dt$$

$$= \int \frac{t^2+2t+1}{(-t^2+t+1)(1+t^2)}\,dt =$$

$$\frac{t^2+2t+1}{(-t^2+t+1)(1+t^2)} = \frac{At+B}{-t^2+t+1} + \frac{Ct+D}{1+t^2}$$

$$\int \frac{\sin^3 x}{\cos^2 x+1}\,dx = \int \frac{\sin^2 x\,\sin x\,dx}{\cos^2 x+1} = \int \frac{(1-\cos^2 x)\sin x}{\cos^2 x+1}\,dx = \int \frac{1-t^2}{1+t^2}(-dt) =$$

$\cos x = t$

$-\sin x\,dx = dt$

$\sin x\,dx = -dt$

$$= \int -\frac{(t^2-1)}{t^2+1}(-dt) = \int \frac{t^2-1}{t^2+1} = \int 1dt - 2\int \frac{1}{1+t^2}\,dt = t + \operatorname{arctg} t + C$$

$$\begin{aligned} &t^2-1 : t^2+1 = 1 \\ &\underline{-t^2\mp 1} \\ &\quad -2 \end{aligned} \qquad \frac{t^2-1}{t^2+1} = 1-\frac{2}{t^2+1}$$

2

$$\int \ln^2(\sin x)\cot g\,x\,dx = \int \ln^2(\sin x)\frac{\cos x}{\sin x}\,dx = \int (\ln^2 t)\frac{1}{t}\,dt = \int z^2\,dz =$$

$\sin x = t$ $\quad$ $\ln t = z$

$\cos x\,dx = dt$ $\quad$ $\frac{1}{t}dt = dz$

$$\int \frac{1}{x\sqrt{1+x^2}}\,dx \cdot \frac{x}{x} = \int \frac{x}{x^2\sqrt{1+x^2}}\,dx = \ldots = \frac{1}{2}\int \frac{2t}{(t^2-1)t}\,dt =$$

$1+x^2 = t^2$

$2x\,dx = 2t\,dt$

$x^2 = t^2 - 1$

$$= \frac{1}{(t-1)(t+1)}\,dt = -\int \frac{1}{1-t^2}\,dt = -\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| dt$$

URČITÝ INTEGRÁL

DEF: MAJME INTERVAL $\langle a,b\rangle$. NECH $x_0, x_1, x_2, \ldots x_m$ SÚ BODY, ŽE $a = x_0 < x_1 < x$ $\ldots < x_m = b$

TO JE DELENIE D INTERVALU $\langle a,b\rangle$

x_i SÚ DELIACE BODY

$\langle x_0,x_1\rangle\langle x_1,x_2\rangle \ldots \langle x_{m-1},x_m\rangle$ (ČIASTOČNÉ INTERVALY)

$\langle x_3,x_4\rangle = \Delta x$

DĹŽKU INT. $\langle x_{i-1},x_i\rangle$ OZNAČUJEME Δx_i

MAX $\{\Delta x_1, \Delta x_2, \ldots, \Delta x_m\}$ JE NORMA DELENIA D

~~NECH JE POSTUPNOSŤ~~ ZO $\forall$ DELENÍ DELENIA D $\|D\|$ NORMA DELENIA A JE TO MAXIMUM $\forall$ DELENÍ $\{D_m\}$

POSTUPNOSŤ DELENÍ $\{D_m\}$ JE FUNKCIA DEFINOVANÁ NA PRIRODZENÝCH ČÍSLACH

AK $\lim \|D_m\| = 0$, TAK JE POSTUPNOSŤ DELENÍ NORMÁLNA.

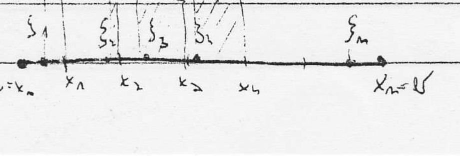
$D_1\ \langle x_0,x_1 \ldots, x_m\rangle$ $\quad \|D_1\|$

$D_2\ \langle x_1,x_2 \ldots x_n\rangle$ $\quad \|D_2\|$

$D_3\ \langle k_1,k_2 \ldots k_m)$ $\quad \|D_3\|$

NECH f JE DEFINOVANÁ A OHRANIČENÁ NA INTERVALE $\langle a,b\rangle$ A $D = \{x_1, x_2, x_3 \ldots x_n$ ~~BODOV~~ JE DELENIE.

V $\forall$ INTERVALE $\langle x_{i-1},x_i\rangle$ SA VYBERIE BOD $f\ \xi_i$



3

Mx 5
SF 15
MG 10,14
MF 15
RL 17

3 11

SOSTIT. INT. TRIGONOM. F

$\operatorname{tg}\frac{x}{2}=t \qquad \sin x=\frac{2t}{1+t^2} \qquad dx=\frac{2}{1+t^2}\,dt$

$\frac{x}{2}=\operatorname{arctg} t \qquad \cos x=\frac{1-t^2}{1+t^2}$

$\frac{1}{2}dx=\frac{1}{1+t^2}\,dt$

$$\int \frac{1}{\sin^3 x}\,dx=\int \frac{1}{\left(\frac{2t}{1+t^2}\right)^3}\,\frac{2}{1+t^2}\,dt=\int \frac{1}{\frac{8t^3}{(1+t^2)^3}}\cdot\frac{2}{1+t^2}\,dt=\frac{1}{4}\int\frac{(1+t^2)^2}{t^3}\,dt=$$

$$=\frac{1}{4}\int\frac{1+2t^2+t^4}{t^3}\,dt=\frac{1}{4}\int\left(\frac{1}{t^3}+\frac{2}{t}+t\right)dt=\frac{1}{4}\left(\frac{t^{-2}}{-2}+2\ln|t|+\frac{t^2}{2}\right)+c$$

$$\int\frac{1}{\sin^3 x}\,dx=\int\frac{\sin x}{\sin^4 x}\,dx=\int\frac{\sin x}{(1-\cos^2 x)^2}\,dx=-\int\frac{1}{(1-t^2)^2}\,dt$$

$\cos x=t \qquad \sin x\,dx=dt \qquad \frac{1}{(1-t^2)^2}=\frac{At+B}{(1-t^2)^2}+\frac{Ct+D}{1-t^2}$

$-\sin x\,dx=dt$

$y=x^3+3x^2-9x+14$

$3x^2+6x-9=0$

$3(x^2-2x-3)=0$

$x^2-2x-3=0$

$$\int e^x\sin x\,dx=\left|\begin{matrix}f=e^x & g'=\sin x\\ f'=e^x & g=-\cos x\end{matrix}\right|=-e^x\cos x+\int e^x\cos x\,dx=\left|\begin{matrix}f=e^x & g'=\cos x\\ f'=e^x & g=\sin x\end{matrix}\right|=-e^x\cos x+\left(e^x\sin x-\int e^x\sin x\,dx\right)$$

$$2\int e^x\sin x=-e^x\cos x+e^x\sin x \Rightarrow \int e^x\sin x=\frac{-e^x\cos x+e^x\sin x}{2}+c$$

$$\int\cos^2 x\,dx=\left|\begin{matrix}f=\cos x & g'=\cos x\\ f'=-\sin x & g=\sin x\end{matrix}\right|\overset{*}{=}\cos x\sin x-\int-\sin^2 x\,dx=\left|\begin{matrix}f=\sin x & g'=\sin x\\ f'=\cos x & g=-\cos x\end{matrix}\right|=\sin x\cos x+$$

$$+\left(\cancel{-\sin x\cos x}+\cancel{\int\cos^2 x\,dx}\right)=$$

$$2\int\cos^2 x=\sin x\cos x$$

$*$ $$=\cos x\sin x+\int(1-\cos^2 x)\,dx=\cos x\sin x+x-\int\cos^2 x\,dx \Rightarrow \int\cos^2 x\,dx=\frac{\cos x\sin x+x}{2}$$

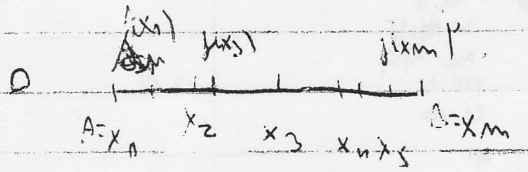
$$\int\frac{1}{\ldots\sin x-7\sin x-7}\,dx=\ldots=\int\frac{1}{4t-7}\,dt=\frac{1}{4}\int\frac{4}{4t-7}=\frac{1}{4}\ln|4t-7|+c$$

$\operatorname{tg}\frac{x}{2}=t\ldots$

4

30.10.08!

10.11.



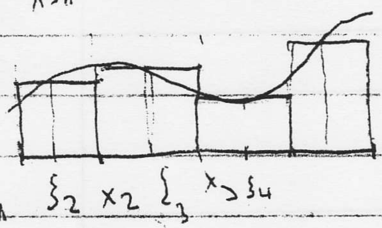
$\Delta x_1, \Delta x_2 \ldots \Delta x_m$

$\langle a, b \rangle$

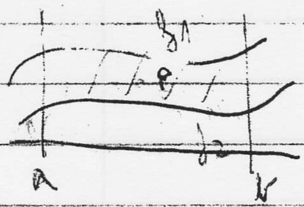
NORMA $\|D\| = \max\{\Delta x_1, \Delta x_2, \Delta x_3, \ldots, \Delta x_m\}$

INTEGRÁLNY SÚČET $S_D(D)$ ZÁVISÍ OD DELENIA A OD VÝBERU BODOV ξ_i

$S_D(D) = \sum_{i=1}^{m} f(\xi_i)\,\Delta x_i$



$\int_a^b f(x)\,dx = [F(x)]_a^b$



$\int_a^b (f_1 - f_2)\,dx = P$

13.11.

VETA: AK JE f SPOJITÁ NA INTERVALE $\langle a, b\rangle$ A VIEME NÁJSŤ FUNKCIU F, KTORÁ JE SPOJITÁ NA $\langle a, b\rangle$ A JE F PRIMITÍVNA K f NA (a, b), POTOM HODNOTU $\int_a^b f(x)\,dx$ MOŽNO VYPOČÍTAŤ PODĽA VZORCA $\int_a^b f(x)\,dx = F(b) - F(a)$ – LEIBNITZOV-NEWTONOV VZOREC

F JE PRIMIT. K $f(x)$, T.J. $F(x) = \int f(x)\,dx \Rightarrow \int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$

$\int_0^{\frac{\pi}{2}} \cos x = [\sin x]_0^{\frac{\pi}{2}} = 1$

OBSAH ROVINNÝCH ÚTVAROV

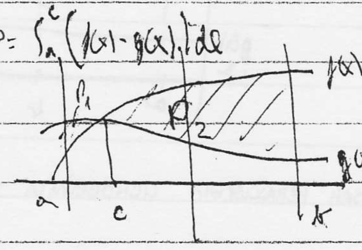
NECH $f(x)$ A $g(x)$ SÚ SPOJITÉ NA $\langle a, b\rangle$ A NA (a, b) JE $g(x) < f(x)$

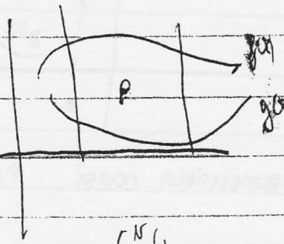
MNOŽINA BODOV $A=[x, y]$, PRE KTORÉ PLATÍ $a \le x \le b$, $g(x) \le y \le f(x)$ JE ELEMENTÁRNA OBLASŤ URČENÁ FUNKCIAMI f A g A INTERVALOM $\langle a, b\rangle$.

AK JE $g(x)=k$ KONŠTANTNÁ, POTOM ELEMENTÁRNA OBLASŤ SA NAZÝVA KRIVOČIARY LICHOBEŽNÍK, PRE KTORÝ PLATÍ

$a \le x \le b \quad k \le y \le f(x)$

20.11. $P = P_1 + P_2$ $\quad P = \int_a^c (f(x) - g(x))\,dx$

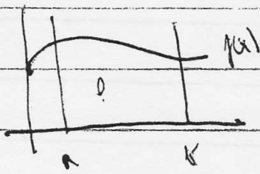




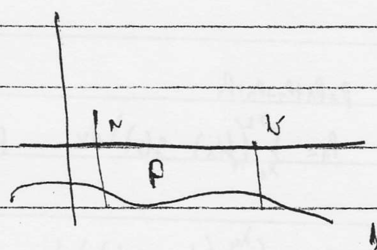
$P = \int_a^b (f(x) - g(x))\,dx$

$g(x)$ je primit. k $f(x)$ $\qquad \int_a^b f(x)\,dx = [g(x)]_a^b$

$g(x) = \int f(x)\,dx$



$g(x) = 0 \qquad P = \int_a^b f(x)\,dx$

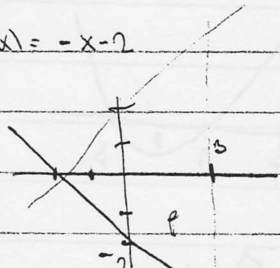


$f(x) = -(x^3+2)$

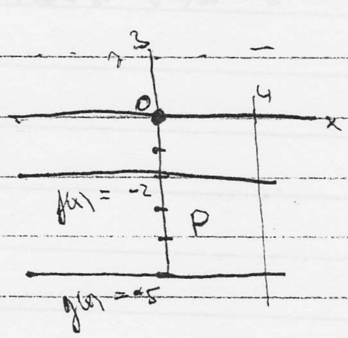
$P = \int_a^b [0 - f(x)]\,dx$

$g(x) = 0$

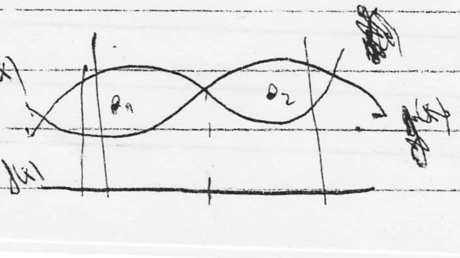
$f(x) = -x-2$



$P = \int_0^3 0 - (-x-2)\,dx = \int_0^3 (x+2)\,dx$



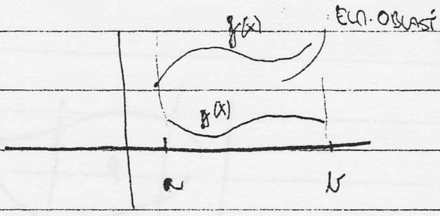
$P = \int_0^4 [-2-(-5)]\,dx = [3x]_0^4 = 12$

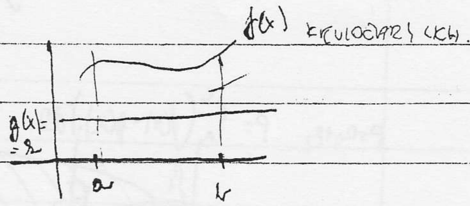


$P_1 = \int_a^c [f(x) - g(x)]\,dx$

$P_2 = \int_c^b [g(x) - f(x)]\,dx$

6



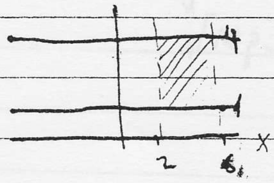


ELEMENTÁRNA OBLASŤ $P = \int_a^b (f(x) - g(x))\,dx$. OBSAH KRIVOČIAREHO LICHOBEŽNÍKA JE $\int_a^b |f(x) - 2|\,dx$

$f(x) = 4$

$g(x) = 1$

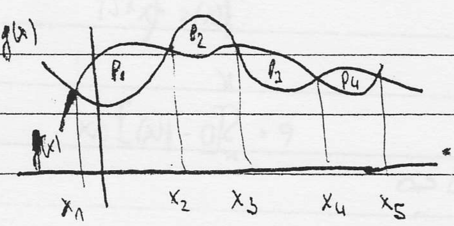
INT $\langle 2, 6 \rangle$



$\int_2^6 (4-1)\,dx = 18 - 6 = 12$

⑫ NÁJSŤ VEĽKOSŤ PLOCHY MEDZI KRIVKAMI X-OVOU OSOU $(g(x))$ A FUNKCIOU $f(x) = x^2 - 4x + 3$

$g(x) = 0$



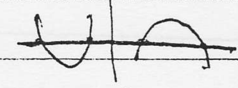
$P = P_1 + P_2 + P_3 + P_4$

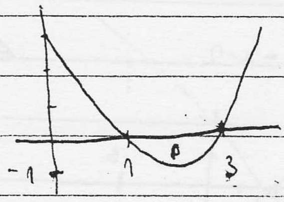
$P_1 = \int_{x_1}^{x_2} (f(x) - g(x))\,dx$ $\quad P_2 = \int_{x_2}^{x_3} (g(x) - f(x))\,dx$

$P_3 = \int_{x_3}^{x_4} (f(x) - g(x))\,dx$ $\quad P_4 = \int_{x_4}^{x_5} (\overset{\text{HORNÁ}}{g(x)} - f(x))\,dx$

$x^2 - 4x + 3 = 0 \quad D = 4 \quad x_1, x_2 = \frac{4 \pm 2}{2} \begin{matrix} 3 \\ 1 \end{matrix}$

$D > 0$ - ~~DA~~.





$f(2) = -1$

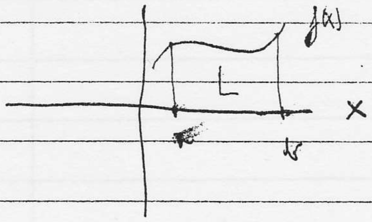
$f(0) = 3$

$P = \int_1^3 0 - (x^2 - 4x + 3)\,dx = \int_1^3 -x^2 + 4x - 3 = \left[-\frac{x^3}{3} + \frac{2x^2}{2} - 3x \right]_1^3$

$= \cancel{3}(-9 + 18 - \cancel{27}) - (-\frac{1}{3} - 1) = \frac{4}{3}$

OBJEM ROTAČNÝCH TELIES

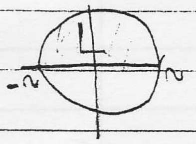
NECH $f(x)$ JE SPOJITÁ NA $\langle a, b \rangle$ A $f(x) \geq 0$ PRE $x \in \langle a, b \rangle$. L = KRIVOČIARY LICHOBEŽNÍK $L = \{x, y\}: x \leq a \leq b$, $0 \leq y \leq f(x)$



TELESO, KTORÉ OPÍŠE KRIVOČIARY LICHOBEŽNÍK L ~~SA NAZÝ~~ PRI ROTÁCII OKOLO OSI X NAZÝVAME ROTAČNÝM TELESOM, URČENÝM FUNKCIOU f A INTERVALOM $\langle a, b \rangle$.

OBJEM V ROTAČNÉHO TELESA T JE $V = \pi \int_a^b f^2(x)\,dx$

PR. OBJEM KRUŽNICE O POLOMERE r



$x^2 + y^2 = r^2$

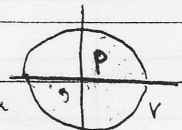
$y = \sqrt{r^2 - x^2}$

$L = \sqrt{r^2 - x^2}$

$$V = \pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx = \pi \int_{-r}^{r} (r^2 - x^2)\,dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} =$$

$$= \pi \left[\left(r^3 - \frac{r^3}{3}\right) - \left(r^2(-r) - \frac{-r^3}{3}\right) \right] = \pi \left(\frac{2}{3} r^3 + \frac{2}{3} r^3 \right) = \frac{4}{3} \pi r^3$$

PR.



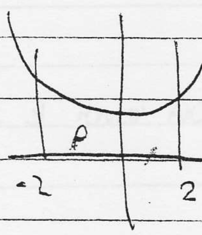
$x^2 + y^2 = r^2$

$$\pi \int_0^r \left(\sqrt{r^2 - x^2}\right)^2 dx = \pi \int_0^r (r^2 - x^2)\,dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_0^r = \pi \left[\left(r^3 - \frac{r^3}{3} \right) \right]_0^r = 22.M.$$

$$= \pi \left[\left(r^3 - \frac{r^3}{3} \right)^0 - \left(r^2 \cdot 0 - \frac{0^3}{3} \right) \right] = \pi \left[\frac{3r^3 - r^3}{3} \right] = \frac{2}{3} \pi r^3$$

PR. $-2 \leq x \leq 2$

$0 \leq y \leq x^2 + 2$



$$V = \pi \int_{-2}^{2} (x^2 +)^2\,dx = \pi \left[\frac{x^5}{5} + \frac{4}{3} x^3 + 4x \right]_{-2}^{2} =$$

$$= \left(\frac{32}{5} + \frac{32}{3} + 8 \right) - \left(-\frac{32}{5} - \frac{32}{3} - 8 \right) = \ldots$$

PR. $y = x^2$

$y = \sqrt{x}$

$x \in \langle 0, 1 \rangle$

$$V = \pi \int \left((\sqrt{x})^2 - (x^2)^2 \right) dx = \pi \int_0^1 (x - x^4)\,dx = \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \left(\frac{1}{2} - \frac{1}{5} \right) - (0 - 0) = \pi \frac{3}{10}$$

$\boxed{\text{PR. - DĹŽKA KRIVKY}}$

$$S = \int_a^b \sqrt{1 + (f'(x))^2}\,dx \qquad f(x) = 2\sqrt{x} \quad \langle 0, 1 \rangle$$

$$f'(x) = 2 \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}}$$

$$S = \int_0^1 \sqrt{1 + \left(\frac{1}{\sqrt{x}} \right)^2} = \int_0^1 \sqrt{1 + \frac{1}{x}}$$

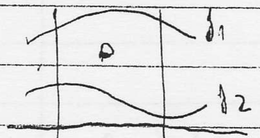
URČ. INT + SPR.

(8)

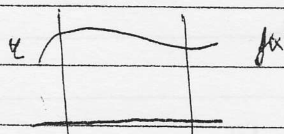
35954 35308

24.11.

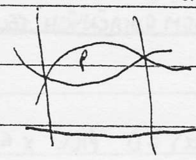
ROTAČNÉ OBJEMY



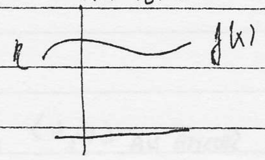
ROTAČNÉ PLOCHY



VEĽKOSŤ ROVINNÝCH PLÔCH



DĹŽKA KRIVKY



VYUŽITIE URČ. INT.

$\int_a^b \sqrt{a^2+x^2}\,dx$

$\sqrt{a^2+x^2} = x+t = \frac{a^2-t^2+2t^2}{2t} = \frac{a^2+t^2}{2t}$

$a^2+x^2 = (x+t)^2$

$a^2+\cancel{x^2} = \cancel{x^2}+2xt+t^2$

$x = \frac{a^2-t^2}{2t} \qquad dx = \frac{-4t^2-(a^2-t^2)\cdot 2}{4t^2} = \frac{-4t^2-2a^2+2t^2}{4t^2} = \frac{-2(t^2+a^2)}{4t^2} = -\frac{1}{2}\frac{t^2+a^2}{t^2}$

$\int \sqrt{a^2+x^2}\,dx = \int \frac{a^2+t^2}{2t}\cdot\left(-\frac{1}{2}\right)\frac{a^2+t^2}{t^2} = -\frac{1}{4}\int\frac{(a^2+t^2)^2}{t^3} = -\frac{1}{4}\int\frac{a^4+2a^2t^2+t^4}{t^3}\,dt = \frac{1}{4}a^4\int\frac{1}{t^3}\,dt -$

$-\frac{1}{4}2a^2\int\frac{1}{t}\,dt - \frac{1}{4}\int t\,dt$

$t = \sqrt{a^2+x^2}-x$

OBJEM ROTAČ. TELESA:

NECH $f(x), g(x)$ SÚ SPOJITÉ NA $\langle a,b\rangle$ A PRE $x\in\langle a,b\rangle$ JE $g(x)\le y \le f(x)$

AK TELESO JE DANÉ $[x,y]$: $a\le x\le b$

$0\le y$

POTOM OBJEM ROTAČ. TELESA JE $V=\pi\int_a^b (f(x)-g(x))\,dx$

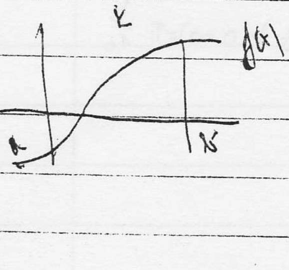
AK $g(x)=0$ $\quad V=\pi\int f^2(x)\,dx$

$\frac{a+b}{c} = \frac{a}{c}+\frac{b}{c}$

DĹŽKA ROVINNEJ KRIVKY

NECH KRIVKA k JE GRAFOM FUNKCIE $f(x)$ NA $x\in\langle a,b\rangle$. DĹŽKA KRIVKY k JE $s=\int_a^b\sqrt{1+(f'(x))^2}\,dx$

$f(x)$ MÁ SPOJITÚ DERIVÁCIU NA INT. $\langle a,b\rangle$



9

1.12. $f(x,y) = x^y$

$f'_x = yx^{y-1}$

$f'_y = x^y \ln x$

$f(x,y) = \left(\frac{y}{z}\right)^x$

$f'_x = \left(\frac{y}{z}\right)^x \ln\frac{y}{z}$

$f'_y = xy^{x-1} \cdot (-xz^{-x-1})$

$f = x^y$

$f'_x = x^y = e^{y\ln x} = e^{y\ln x}(y\ln x)' = x^y \cdot \left(y\frac{1}{x}\right) = x^y\left(y\frac{1}{x}\right) = yx^y \cdot \frac{1}{x} = yx^{y-1}$

$f'_y = e^{y\ln x}(y\ln x)' = e^{y\ln x}\ln x = x^y \ln x$

$f(x,y) = \ln\left(\frac{x+y}{x-y}\right)$

$f'_x = \frac{1}{\frac{(x+y)}{x-y}} \cdot \frac{1(x-y)-(x+y)\cdot 1}{(x-y)^2} = \frac{\cancel{(x-y)}}{(x+y)} \cdot \frac{-2y}{(x-y)^2} = \frac{-2y}{(x+y)(x-y)} = \frac{-2y}{x^2-y^2}$

$f(x,y) = (\ln x)^{\cos y} = e^{\cos y \ln\ln x} = e^{\cos y(\ln(\ln x))}\left(\cos y \ln(\ln x)\right)' = (\ln x)^{\cos y} \cdot \cos y \frac{1}{\ln x} \cdot \frac{1}{x}$

TEORIE – URČ. POMOCOU NEURČ.

– URČ. INT. DEFINOVAŤ PLOCHAMI

$f = (\ln x)^{\cos y}$

4.12. $f'_x = \left(e^{\cos(\ln(\ln x))}\right)\left(\cos y(\ln(\ln x))\right)' = (\ln x)^{\cos y} \cdot \left(\cos y \frac{1}{\ln x}\frac{1}{x}\right)$

$f'_y = \left(e^{\cos y \ln(\ln x)}\right)' = e^{\cos y \ln(\ln x)}\left(\cos y \ln(\ln x)\right)' = (\ln x)^{\cos y} \cos y \ln(\ln x)$

$f(x,y,z) = 2^{xyz}$

$f'''_{xzy} =$ ~~...~~

$f'_x =$ ~~...~~ $2^{xyz} \ln 2 (xyz)' = 2^{xyz} \ln 2 \cdot yz = \ln 2 \cdot 2^{xyz} yz$

$f'_{xz} = \ln 2^{xyz} \ln 2 \cdot (xyz)' yz) + (2^{xyz} y)$

Délka křivky

$f(x)=x^2$ $\qquad x\in\langle 0,1\rangle$ $\qquad f'(x)=2x$

$$S=\int_a^b \sqrt{1+(f'(x))^2}\,dx = \int_0^1\sqrt{1+(2x)^2}\,dx = \int_0^1\sqrt{1+4x^2}\,dx = \int \frac{t^2+1}{2t}\cdot\frac{-t}{2t}\,dt =$$

$$\sqrt{1+4x^2}=t+2x = t+2\,\frac{1-t^2}{4t} = \frac{4t^2+2-2t^2}{4t} = \frac{\cancel{2}(t^2+1)}{\cancel{2}\,2t}=\frac{t^2+1}{2t}$$

$$1+\cancel{4x^2} = t^2+4xt+\cancel{4x^2}$$

$$4xt=1-t^2 \Rightarrow x=\frac{1-t^2}{4t}$$

$$4t\,dx=-2t\,dt$$

$$dx=-\frac{2t}{4t}\,dt=-\frac{t}{2t}\,dt$$

$$=-\frac14\int\frac{\cancel{t}(t^2+1)}{t^{\cancel{2}}}\,dt = -\frac14\int\frac{t^2+1}{t}\,dt=-\frac14\int(t+t^{-1})\,dt = -\frac14\left(\frac{t^2}{2}+\ln t\right)=$$

$=$

Veta: Ak je f spojitá na určitom intervale $\langle a,b\rangle$ a vieme nájsť funkciu F, kt. je tiež spojitá na $\langle a,b\rangle$ a F je primitívna k f na (a,b) $\Rightarrow$ hodnotu $\int_a^b f(x)\,dx$ možno vyčísliť podľa vzorca $\int f(x)\,dx = F(b) - F(a)$

↑ toto je Leibnitzov - Newtonov vzorec

$\int_a^b f(x)\,dx$ F je primitív k $f(x)$ t.j. je $F(x) = \int f(x)\,dx$

počítam $\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$

$$\int_1^2 x^6\,dx = \left[\frac{x^7}{7}\right]_1^2 = \frac{2^7}{7} - \frac{1^7}{7} = \frac{128}{7} - \frac{1}{7} = \frac{1}{7}(128-1) = \frac{127}{7}$$

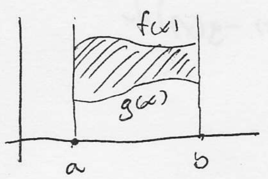
$$\int_0^{\frac{\pi}{2}} \cos x\,dx = \left[\sin x\right]_0^{\frac{\pi}{2}} = 1 - 0 = 1 \quad \int \cos x\,dx = \sin x = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$

OBSAH ROVINNÝCH ÚTVAROV

nech $f(x)$ a $g(x)$ sú spojité na $\langle a,b\rangle$ a na (a,b) je $g(x) < f(x)$

množinu bodov $A = (x,y)$ pre ktoré platí $a \le x \le b$, $g(x) \le y \le f(x)$ je elementárna oblasť určená funkciami f a g a intervalom $\langle a,b\rangle$

Ak je $g(x) = k$ konštantná, potom elementárna oblasť sa nazýva krivočiarym lichobežníkom, pričom platí $a \le x \le b$, $k \le y \le f(x)$



Elementárna oblasť $P = \int_a^b (f(x) - g(x))\,dx$

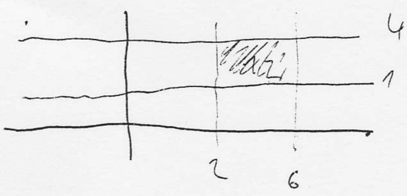
)

$f(x) = 4$

$g(x) = 1$

interval $\langle 2,6\rangle$

$$\int_2^6 (4-1)\,dx = \int_2^6 3\,dx = \left[3x\right]_2^6$$



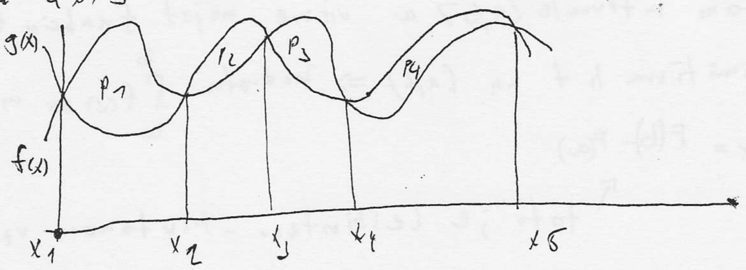
$\int 3\,dx = 3x$

11. Nájst veľkosť plochy medzi krivkou x-ovou osou $-g(x)$ a funkcou 2

$f(x) = x^2 - 4x + 3$

$g(x) = 0$

$P = P_1 + P_2 + P_3 + P_4$



$P_1 = \int_{x_1}^{x_2} (f(x) - g(x))\,dx \qquad P_2 = \int_{x_2}^{x_3} (g(x) - f(x)) \qquad P_3 = \int_{x_3}^{x_4} (f(x) - g(x))\,dx$

$P_4 = \int_{x_4}^{x_5} (g(x) - f(x))\,dx$

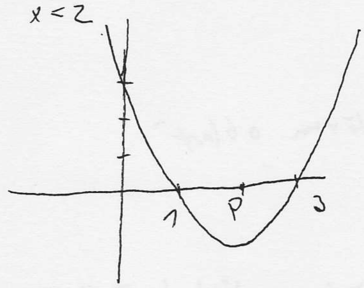
$x^2 - 4x + 3 = 0 \qquad D = 16 - 12 = 4 \qquad x_1 = \dfrac{4 \pm \sqrt{4}}{2} = \dfrac{4 \pm 2}{2} = \begin{cases} 3 \\ 1 \end{cases}$

$D > 0$ - má dve reálne riešenia

$f'(x) = 2x - 4 = 0$

$2x = 4$

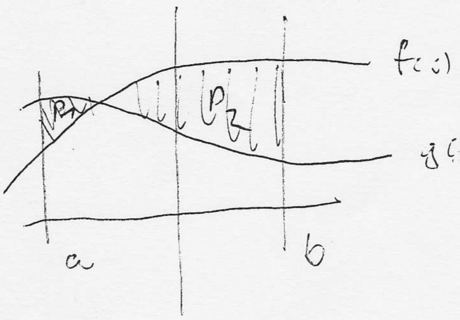
$x = 2$

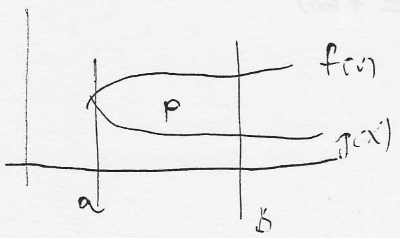


$P = \int_1^3 [0 - (x^2 - 4x + 3)]\,dx = \int_1^3 (-x^2 + 4x - 3)\,dx = \dfrac{-x^3}{3} + \dfrac{\cancel{4}^{2}x^2}{\cancel{2}} - 3x =$

$= \left[-\dfrac{x^3}{3} + 2x^2 - 3x\right]_1^3 = \left(-9 + 18 - 9\right) - \left(-\dfrac{1}{3} + 2 - 3\right) =$

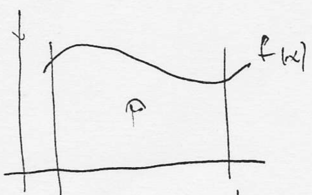
$0 - \left(-1 - \dfrac{1}{3}\right) - \left(-\dfrac{4}{3}\right) = \dfrac{4}{3}$





$P = \int_a^b (f(x) - g(x))\,dx$

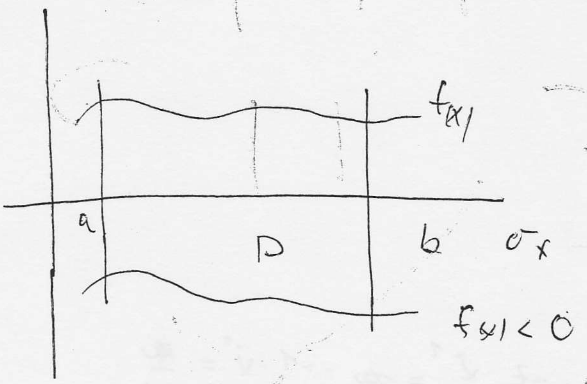
$g(x) = 0$.

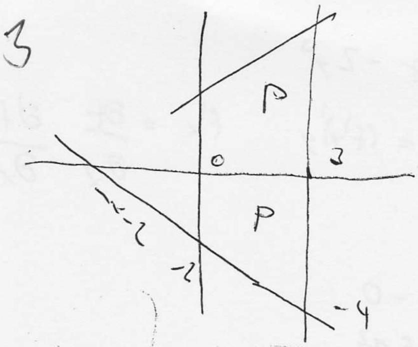


$g(x) = \int f(x)\,dx$

$g(x)$ primit. k $f(x)$ $\quad \int_a^b f(x)\,dx = \left[g(x)\right]_a^b$

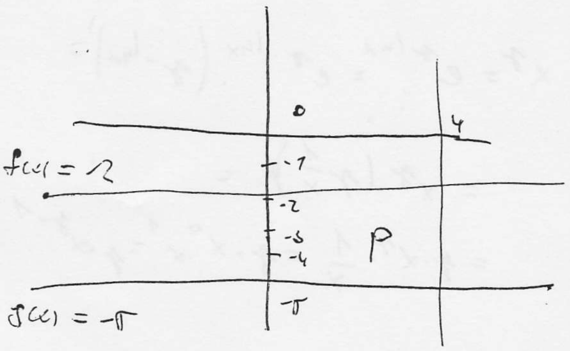
$P = \int (0 - f(x))\,dx$ $\qquad f(x) = -(x^3+2)$ $\qquad 3$





$f(x) = -x-2$

$\int_0^3 0-(-x-2) = \int_0^3 x+2$



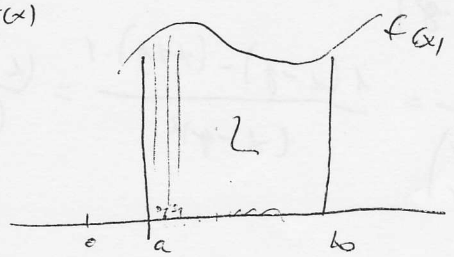
$P = \int_0^4 (-2-(-5))\,dx = \int_0^4 (-2+5)\,dx =$

$= \int_0^4 3\,dx = [3x]_0^4 = 3\cdot 4 - 3\cdot 0 = 12 - 0 = \underline{12}$

$\boxed{\text{V rotač. telies}}$

nech f je spoj. na $\langle a,b\rangle$ a $f(x) \geq 0$

krivočiary lich. $L = \{x, y\}$ $x \leq a \leq b$, $0 \leq y \leq f(x)$

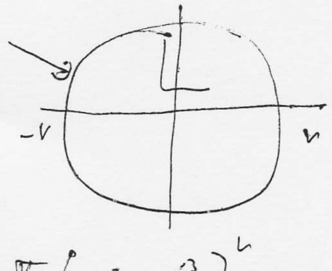


teleso, kt. opíše krivočiary lichobežník L ktorý rotuje okolo osi x nazývame rotačným telesom určeným funkciou f a intervalom $\langle a,b\rangle$

Objem V rotačného telesa T f je $V = \pi \int_a^b f^2(x)\,dx$

Pr. V o polomere r

$\sqrt{r^2-x^2}$



$x^2+y^2=r^2$

$y = \sqrt{r^2-x^2}$

$V = \pi \int_{-r}^{r} (\sqrt{r^2-x^2})^2\,dx = \pi \int_{-r}^{r} (r^2-x^2)\,dx =$

$\int (r^2-x^2)\,dx = r^2x - \frac{x^3}{3}$

$= \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(\left(r^2\cdot(-r) - \frac{(-r)^3}{3}\right)\right)\right] = \pi\left(\left(\frac{2}{3}r^3\right) - \left(-r^3 + \frac{r^3}{3}\right)\right) =$

$= \pi\left(\frac{2}{3}r^3 - \left(-\frac{2}{3}r^3\right)\right) = \pi\left(\frac{2}{3}r^3 + \frac{2}{3}r^3\right) = \underline{\frac{4}{3}\pi r^3}$

$(x,y) = 3x^3 + 5x^2y - 2y^3$ 4

f'_x ; f''_x, $f''_{xy} = (f'_x)'_y$ $\quad f'_x = \frac{\partial f}{\partial x} \quad \frac{\partial^2 f}{\partial x^2}$

f'_y

$f'_x = 9x^2 + 10x \cdot y - 0$

$f'_y = 0 + 5x^2 - 6y^2$

$f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

$x \cdot \frac{1}{y} + 0$

$f'_x = \frac{1}{y} + 0$

$-\frac{z}{x} = -z \cdot \frac{1}{x} \cdot -z \cdot x^{-1} = z \cdot -1 \cdot x^{-2} = \frac{z}{x^2}$

$-\frac{z}{x} \quad f'(x) = \frac{0 \cdot x - z \cdot 1}{x^2} = \frac{z}{x^2}$

$f(x,y) = x^y$

$f'_x = y \cdot x^{y-1}$

$f'_y = x^y \cdot \ln x$

$f(x,y) = \left(\frac{y}{z}\right)^x$

$f'_x = \left(\frac{y}{z}\right)^y \cdot \ln \frac{y}{z}$

$f'_y = e^{y \ln x}(y \cdot \ln x)' =$

$= e^{y \ln x} \cdot \ln x =$

$= x^y \cdot \ln x$

$x^y = e^{y \ln x} = e^{y \cdot \ln x} \cdot (y \cdot \ln x)' =$

$= x^y \cdot (y \frac{1}{x}) =$

$= y \cdot x^y \cdot \frac{1}{x} = y \cdot x^y \cdot x^{-1} = y \cdot x^{y-1}$

$f(x,y) = \ln\left(\frac{x+y}{x-y}\right)$

$f'_x = \frac{1}{\left(\frac{x+y}{x-y}\right)} = \frac{1(x-y) - (x+y) \cdot 1}{(x-y)^2} = \frac{(\cancel{x-y})}{(x+y)} \cdot \frac{-2y}{(x-y)^{\cancel{2}}} = -\frac{2y}{(x+y) \cdot (x-y)} = -\frac{2y}{x^2-y^2}$

$f(x,y) = (\ln x)^{\cos y} = e^{\cos y \cdot \ln(\ln x)}$

$= e^{\cos y(\ln(\ln x))} \cdot (\cos y \cdot \ln(\ln x))' =$

$= (\ln x)^{\cos y} \cdot \cos y \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$

$f(x) = (\ln x)^{\cos y} = e^{\cos y \ln(\ln x)}$ 5

$x = e^{\ln x}$ $f(x)g(x)$ $f'_x = (e^{\cos x(\ln(\ln x))}) \cdot (\cos x(\ln(\ln x)))' = (\ln x)^{\cos y} \cdot (\cos y \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$

$(\cos y(\ln(\ln x)) = \cos y (\ln(\ln x))' = \cos y \cdot (\ln(\ln x))' = \cos y \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$

$f'(x) = (e^{\cos y \cdot \ln(\ln x)})' = e^{\cos y \cdot \ln(\ln x)} (\cos y \cdot \ln(\ln x))'$

$(\cos y \cdot \ln(\ln x)) = \cdot \sin y \cdot \ln(\ln x)$

$= (\ln x)^{\cos y} \cdot \cos y \cdot \ln(\ln x)$

$\left(\frac{y}{x}\right)^x =$

$x^y =$

$(x^y)_y = (e^{y \cdot \ln x})'_y = e^{y \cdot \ln x}(y \cdot \ln x)' = e^{y \cdot \ln x} \cdot 1 \ln x$

$(2^{f(x)})' = 2^{f(x)} \cdot \ln 2 \cdot f(x)' = 2$

$f(x,y,z) = 2^{xyz}$

f'''_{xyz} $f'_x = 2^{x,y,z} \cdot \ln 2 \cdot (x,y,z)' = 2^{xyz} \cdot \ln 2 \cdot 1 \cdot y \cdot z$

$f'_{xz} = \ln 2 \cdot (2^{xyz} \cdot (yz))' = \ln 2 (2^{xyz} \ln 2 \{(xyz)' \cdot yz) + (2^{xyz} \cdot y)$

$\boxed{\text{Pv}}$

lT ke deriv. to z

$f(x) = x^2$ $f'(x) = 2x$

$x \in \langle 0,1 \rangle$

$S = \int_a^b \sqrt{1+(f'(x))^2}\,dx = \int_0^1 \sqrt{1+((x)')^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx = \int \frac{t^2+1}{2t} \cdot \frac{-t}{2t}\,dt =$

$\sqrt{1+4x^2} = t + 2x = t + \frac{2 \cdot (1-t^2)}{4t} = \frac{4t^2+2-2t^2}{4t} = \frac{2(t^2+1)}{2 \cdot 4t} = \frac{t^2+1}{2t}$

$1 + \cancel{4x^2} = t^2 + 4xt + \cancel{4x^2}$

$4xt = 1 - t^2 \Rightarrow x = \frac{1-t^2}{4t}$

$4t\,dx = -2t\,dt$

$\hookrightarrow -\frac{1}{4} \int \frac{t(t^2+1)}{t^2}\,dt$

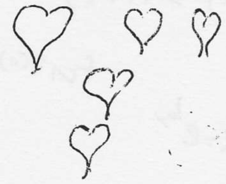
$= -\frac{1}{4}\left(\frac{t^2}{2} + \ln t\right)$

$f(x)$

$\langle a,b\rangle$ $\boxed{P = 2\pi \cdot \int f(x) \cdot \sqrt{1+(f'(x))^2}}$

rotačné plochy

6



$$2\pi \int_a^b f(x) \cdot \sqrt{1+(f'(x))^2}\, dx$$

$k: f(x)$ $\langle a,b\rangle$

$\exists f(x,y,z) = (2xy^2+z^3)^{11}$

$f'(x) = 11 \cdot (2xy^2+z^3)^{10} \cdot 2y^2$

$f(y) = 11 \cdot (2xy^2+z^3)^{10} \cdot 4xy = 44(2xy^2+z^3) \cdot xy$

$2x \cdot 2y$

$f'(x,y) = 11[10(2xy^2+z^3)^9 \cdot 4xy \cdot 2y^2 + (2xy^2+z^3)^{10} \cdot 4y$

$(2^x)' = 2^x \cdot \ln 2$

$f(x,y) = x^y$

$f(x) = y \cdot x^{y-1}$

$f(y)' = x^y \cdot \ln x$

$f(x,y) = y \cdot \sin x + \cos(x-y)$

$f'(x) = y \cdot \cos x + -\sin(x-y)$

$f'(y) = 1 \cdot \sin x + -\sin(x-y) \cdot -1$

$f(x,y) = \ln\left(\frac{x+y}{x-y}\right)$

$f(x) = \frac{1}{\left(\frac{x+y}{x-y}\right)} \cdot \frac{1\cdot(x-y)-(x+y)\cdot 1}{(x-y)^2} = \frac{-2y}{(x-y)\cdot(x+y)}$

$f(x,y) = x^{\frac{1}{y}} \cdot z$

$f'(x) = \frac{1}{y} \cdot x^{\left(\frac{1}{y}-1\right)} \cdot z$

$f'(y) = x^{\frac{1}{y}} \cdot \ln x - \frac{1}{y^2} \cdot z^2$

$f(x,y) = e^{\frac{x}{y}}$

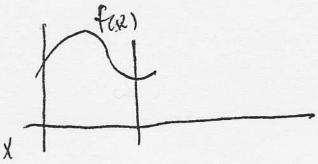
$f'(x) = e^{\frac{x}{y}} \cdot \frac{1}{y}$

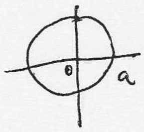
...jaté vlnenie
...nuslátory
...vunt...

Rotač. plochy

$$P = 2\pi \int_a^b f(x)\cdot\sqrt{1+f'(x)^2}\,dx$$

M: $f(x)$, (a,b)





Př) 7

$f(x)=\frac{x^2}{2}$ $0\le x\le \frac{3}{4}$

$f'(x)=\frac{1}{2}\cdot 2x = x$

$P=2\pi\int_0^{3/4}\frac{x^2}{2}\sqrt{1+x^2} = *$

$\sqrt{1+x^2}=x+t \Rightarrow$

$1+\cancel{x^2}=\cancel{x^2}+2xt+t^2$

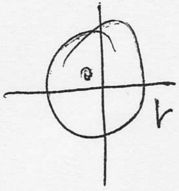
$\frac{1}{2}\cdot\frac{1}{t}-\frac{1}{2}t = \frac{1-t^2}{2t}=x$

$=\frac{1-t^2}{2t}+t = \frac{1-t^2+2t^2}{2t}=\frac{1+t^2}{2t}$

$$* = \int 2\pi\int_0^{3/4}\frac{\left(\frac{1-t^2}{2t}\right)^2}{2}\cdot\frac{1+t^2}{2t}\cdot -\frac{1}{2}\cdot\left(1+\frac{1}{t^2}\right)dt =$$

$$=\pi\int_0^{3/4}\frac{(1-t^2)^2}{4t^2}\cdot\frac{1+t^2}{2t}-\frac{1}{2}\left(\frac{1+t^2}{t^2}\right)dt = 2\pi\int_0^{3/4}\frac{(1-t^2)^2\,(1+t^2)}{8t^3}\cdot -\frac{1}{2t^2}(1+t^2)$$

Př)



$f(x)=\sqrt{r^2-x^2}$, $\langle 0,r\rangle$

$x^2+y^2=r^2$ $f'(x)=\frac{1}{\cancel{2}}\cdot\frac{1}{\sqrt{r^2-x^2}}\cdot -\cancel{2}x = \frac{-x}{\sqrt{r^2-x^2}}$

$$P=2\pi\int_a^b f(x)\cdot\sqrt{1+f'(x)^2}\,dx = \int_0^r\sqrt{r^2-x^2}\cdot\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx=$$

$$=2\pi\int_0^r\sqrt{r^2-x^2}\cdot\sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\,dx = 2\pi\int_0^r\cancel{\sqrt{r^2-x^2}}\cdot\frac{r}{\cancel{\sqrt{r^2-x^2}}}\,dx=2\pi\int_0^r r\,dx=$$

$$=2\pi r[x]_0^r = 2\pi r(r-0)=\underline{2\pi r^2}$$