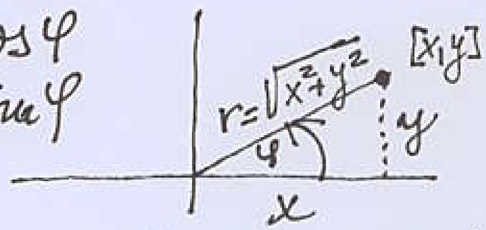
Transformácia dvojného integrálu do polárnych súradníc

polárne súradnice (r,φ), kde $\begin{cases} r>0 \\ \varphi\in\langle 0,2\pi) \end{cases}$

pri zvolenom pravouhlom súr. systéme v rovine je dvojica $\begin{cases} r \text{ ... vzdialenosť bodu od začiatku} \\ \varphi \text{ ... uhol spojnice bodu so začiatkom a kladnej poloosi } x. \end{cases}$

vzťah pravouhlých a polárnych súradníc:

$(r,\varphi) \longrightarrow (x,y)$ ak $x = r\cos\varphi$, $y = r\sin\varphi$



$\cos\varphi = \frac{x}{r}, \ \sin\varphi = \frac{y}{r}$

(začiat.) bod $(0,0)$ má $r=0$

Veta o transf.: Nech D je elementárna oblasť v rovine popísaná nerovnosťami

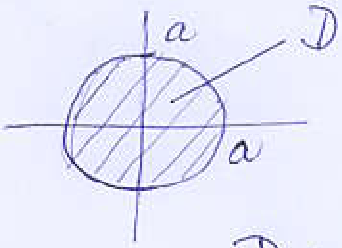
$$D: \left.\begin{array}{c} \alpha \le \varphi \le \beta \\ 0 \le h_1(\varphi) \le r \le h_2(\varphi) \end{array}\right\} \begin{array}{l} 0 \le \beta-\alpha \le 2\pi \\ h_1, h_2 \text{ ... funkcie spojité na } \langle\alpha,\beta\rangle. \end{array}$$

Nech funkcia $f \subseteq R^2\times R$ je spojitá na D.

Potom
$$\iint_D f(x,y)\,dx\,dy = \int_\alpha^\beta \left(\int_{h_1(\varphi)}^{h_2(\varphi)} f(r\cos\varphi, r\sin\varphi)\,\underbrace{r}_{|J|}\,dr \right) d\varphi$$

kde
$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r}, & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r}, & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi, & -r\sin\varphi \\ \sin\varphi, & r\cos\varphi \end{vmatrix} = r(\cos^2\varphi + \sin^2\varphi) = r$$
je determinant transformácie.

- Vypočítajte plošný obsah kruhu
- polomere $a>0$.



transf: $x = r\cos\varphi$
$y = r\sin\varphi$

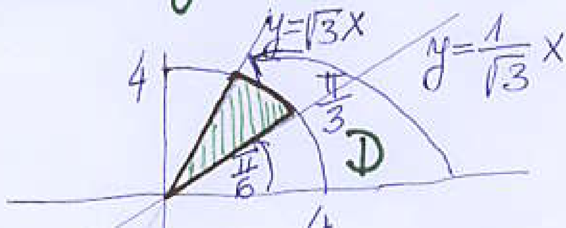
D: $0 \le r \le a$
$0 \le \varphi \le 2\pi$

$|J| = r$

$$\iint_D 1 . dxdy = \int_0^a \left(\int_0^{2\pi} r\, d\varphi \right) dr =$$

$$= \int_0^a r [\varphi]_0^{2\pi} dr = \int_0^a r . 2\pi\, dr = 2\pi \left[\frac{r^2}{2} \right]_0^a = 2\pi \frac{a^2}{2} = \pi a^2$$

- Vypočítajte plošný obsah časti kruhu ohraničenej priamkami $y = \frac{1}{\sqrt{3}} x$, $y = \sqrt{3} x$ pre $x>0$, $y>0$, a kruž. $x^2 + y^2 = 16$.



D: $0 \le r \le 4$
$\frac{\pi}{6} \le \varphi \le \frac{\pi}{3}$ pretože $tg \frac{\pi}{6} = \frac{1}{\sqrt{3}}$, $tg \frac{\pi}{3} = \sqrt{3}$
sú smernice tých 2 priamok

$$C(D) = \iint_D 1\, dxdy =$$

$$= \int_0^4 \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r\, d\varphi \right) dr = \int_0^4 r [\varphi]_{\frac{\pi}{6}}^{\frac{\pi}{3}} dr = \int_0^4 \frac{\pi}{6} r\, dr = \frac{\pi}{6} \left[\frac{r^2}{2} \right]_0^4 = \frac{4\pi}{3}$$

- $\iint_D x^2 \sqrt{x^2+y^2}\, dxdy$, kde D je oblasť z predošlého príkladu

$$= \int_0^4 \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r . r^2 \cos^2\varphi \sqrt{r^2\cos^2\varphi + r^2\sin^2\varphi}\, d\varphi \right) dr,$$ pretože

$x = r\cos\varphi$, $y = r\sin\varphi$, $|J| = r$

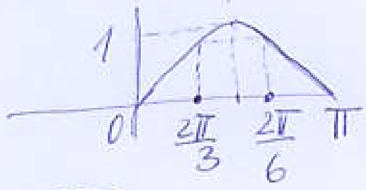
pokračovanie

→ pokračovanie

$$\iint_D x^2\sqrt{x^2+y^2}\,dx\,dy = \int_0^4\Big(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r^4\cdot \underbrace{\cos^2\varphi}_{\frac{1+\cos 2\varphi}{2}}\,d\varphi\Big)dr =$$

$$= \int_0^4 r^4\Big[\tfrac{1}{2}\varphi + \frac{\sin 2\varphi}{4}\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}\,dr =$$

$$= \int_0^4 r^4\Big(\frac{\pi}{12} + \frac{1}{4}\big(\underbrace{\sin\tfrac{2\pi}{3}}_{\frac{\sqrt{3}}{2}} - \underbrace{\sin\tfrac{2\pi}{6}}_{\frac{\sqrt{3}}{2}}\big)\Big)dr =$$

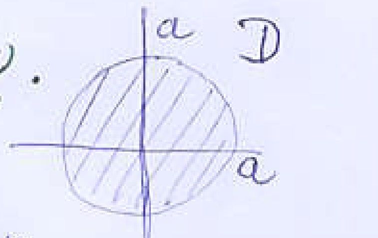


$$= \int_0^4 r^4\cdot\frac{\pi}{12}\,dr =$$

$$= \frac{\pi}{12}\Big[\frac{r^5}{5}\Big]_0^4 = \frac{\pi}{12}\cdot\frac{4^5}{5} = \frac{4^5\cdot\pi}{60} = \frac{4^4\cdot\pi}{15}$$

• ✓ Vypočítajte $\iint_D y^2\sqrt{x^2+y^2}\,dx\,dy$

kde D je kruh o polomere $a>0$.



$$\iint_D y^2\sqrt{x^2+y^2}\,dx\,dy =$$

transf.:
$x = r\cos\varphi$
$y = r\sin\varphi$ $\quad |J| = r$

$0 \le r \le a$, $0 \le \varphi \le 2\pi$ } D

$$= \int_0^a\Big(\int_0^{2\pi} r\cdot r^2\sin^2\varphi\,\underbrace{\sqrt{r^2\cos^2\varphi + r^2\sin^2\varphi}}_{=r}\,d\varphi\Big)dr =$$

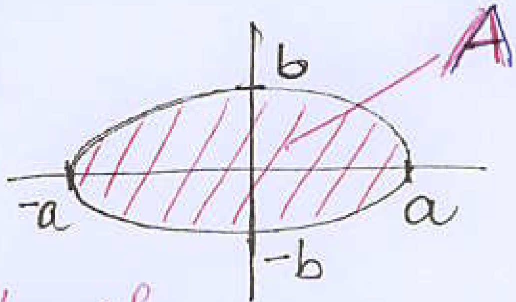
$$= \Big(\int_0^a r^4\,dr\Big)\Big(\int_0^{2\pi}\underbrace{\sin^2\varphi}_{=\frac{1-\cos 2\varphi}{2}}\,d\varphi = \Big[\frac{r^5}{5}\Big]_0^a\Big[\frac{\varphi}{2} - \frac{\sin 2\varphi}{4}\Big]_0^{2\pi} =$$

$$= \frac{a^5}{5}\cdot 2\pi\cdot\frac{1}{2} = \frac{\pi}{5}a^5$$

Elipsa a modifikácia transf. rovníc pomocou polárnych súradníc

✓ Nech $a>0, b>0$ a nech A je (oblasť) množina bodov (x,y) roviny pre ktoré

$$A: \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$$



Zobrazenie ϕ dané transf. rovnicami

$$\phi: \begin{cases} x = a\,r\cos\varphi \\ y = b.r\sin\varphi \end{cases}$$

zobrazí množinu

$$(r,\varphi):\ \left.\begin{array}{l} 0\le r\le 1 \\ 0\le \varphi\le 2\pi \end{array}\right\} A$$

na oblasť A, pretože z transf. rovníc je

$$\frac{x}{a}=r\cos\varphi$$
$$\frac{y}{b}=r\sin\varphi$$

a teda $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=r^2\le 1$ (keďže $0\le r\le 1$)

$$|J|=\begin{vmatrix} \frac{\partial x}{\partial r}, & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r}, & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} a\cos\varphi, & -ar\sin\varphi \\ b\sin\varphi, & br\cos\varphi \end{vmatrix} = abr$$

✓ $$C(A)=\iint_A 1\,dx\,dy=\int_0^1\Big(\int_0^{2\pi} \underbrace{abr}_{|J|}\,d\varphi\Big)dr=$$

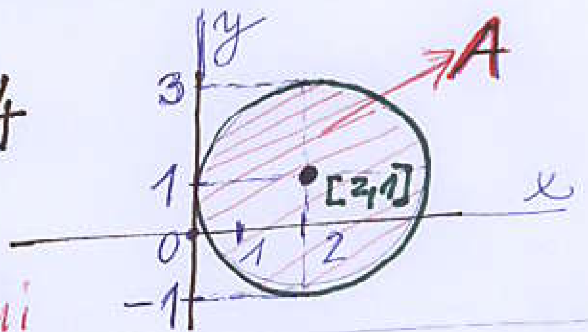
$$=\int_0^1 abr\,[\varphi]_0^{2\pi}\,dr = a.b.2\pi\Big[\frac{r^2}{2}\Big]_0^1=\boxed{\pi a b}$$

je plošný obsah A.

"Posunutá" kružnica a modifikácia transformačných rovníc pomocou polárnych súradníc.

- Ne A je množina bodov (x,y) roviny pre ktoré

$$(x-2)^2+(y-1)^2 \leq 4$$



zobrazenie ϕ dané transformačnými rovnicami

$$\phi: \begin{cases} x = 2 + r\cos\varphi \\ y = 1 + r\sin\varphi \end{cases}$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r}, & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r}, & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi, & -r\sin\varphi \\ \sin\varphi, & r\cos\varphi \end{vmatrix} = r$$

zobrazí množinu (r,φ) : $\left.\begin{matrix} 0 \leq r \leq 2 \\ 0 \leq \varphi \leq 2\pi \end{matrix}\right\} A$

na oblasť A, pretože:

z transf. rovníc je $(x-2)^2 = r^2\cos^2\varphi$

$(y-1)^2 = r^2\sin^2\varphi$

a teda: $(x-2)^2+(y-1)^2 = r^2 \leq 4$ (keďže $0 \leq r \leq 2$)

- $$\iint\limits_{A \text{ (oblasť hore)}} (x-2)\,y\,dx\,dy = \int_0^2 \left(\int_0^{2\pi} r\cos\varphi\,(1+r\sin\varphi) \cdot \underbrace{r}_{|J|}\, d\varphi \right) dr =$$

$$= \int_0^2 \left(\int_0^{2\pi} (r^2\cos\varphi + r^3\cos\varphi\sin\varphi)\, d\varphi \right) dr =$$

$$= \int_0^2 \left[r^2\sin\varphi + r^3\frac{\sin^2\varphi}{2} \right]_0^{2\pi} dr = \int_0^2 (r^2 \cdot 0 + r^3 \cdot 0)\, dr = \underline{\underline{0}}$$

kde: $\begin{cases} A: (x-2)^2+(y-2)^2 \leq 4 \text{ je oblasť hore!} \\ \text{teda: } 0 \leq r \leq 2,\ 0 \leq \varphi \leq 2\pi \text{ (ak } x = 2+r\cos\varphi,\ y = 1+r\sin\varphi) \end{cases}$