Výpočet dvojných integrálov cez elementárne oblasti

(1) A je elementárna oblasť typu $\ll x, y \gg$:

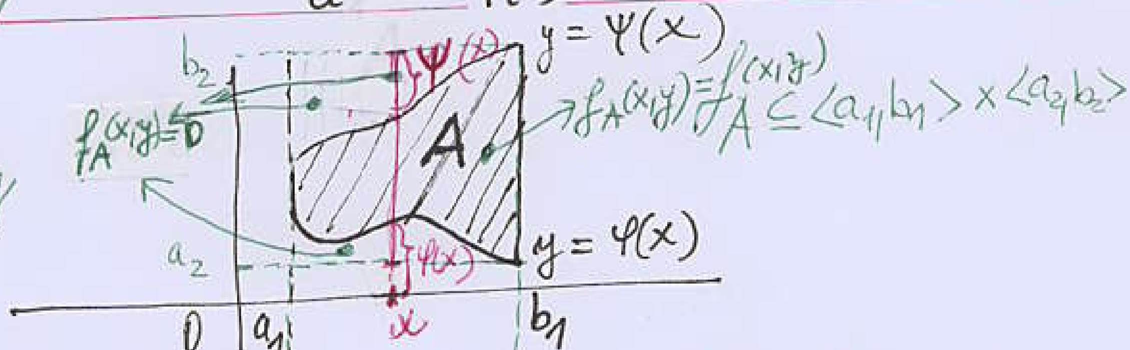
$$A: \left.\begin{array}{l} a_1 \leq x \leq b_1 \\ \varphi(x) \leq y \leq \psi(x) \end{array}\right\} \varphi, \psi \subseteq R \times R \text{ spojité na } \langle a_1, b_1 \rangle$$

$f \subseteq R^2 \times R$ ohraničené a spojitá na A najviac okrem hranice A a konečne veľa vnútorných bodov A

Potom
$$\iint_A f(x,y)\,dx\,dy = \int_a^{b_1} \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy \right) dx$$

pretože:

$$\iint_{\langle a_1,b_1\rangle \times \langle a_2,b_2\rangle} f_A(x,y)\,dx\,dy$$



pri pevne zvolenom $x \in \langle a_1, b_1 \rangle$ je

$$f_A(x,y) = \begin{cases} f(x,y) & \text{pre } x \in A \\ 0 & \text{pre } x \notin A \end{cases} \Rightarrow \int_{a_2}^{b_2} f_A(x,y)\,dy = \int_{a_2}^{\varphi(x)} 0\,dy + \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy + \int_{\psi(x)}^{b_2} 0\,dy$$

$$= \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy$$

(2) A: $a_2 \leq y \leq b_2$

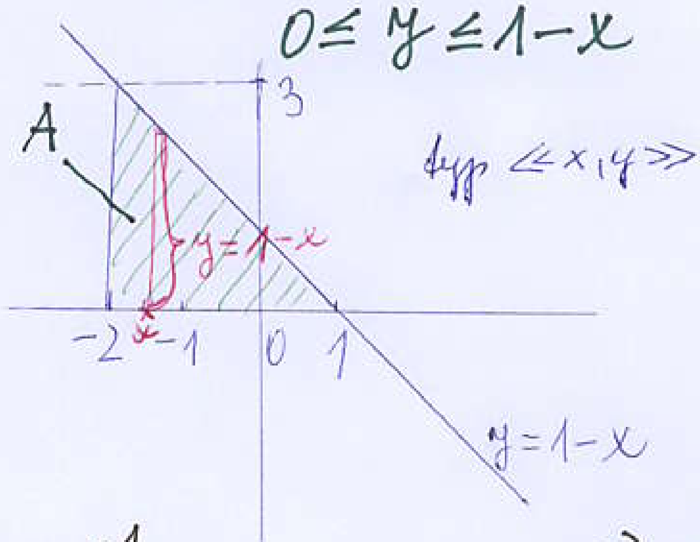
typ $\ll y, x \gg$ $\alpha(y) \leq x \leq \beta(y)$

$$\iint_A f(x,y)\,dx\,dy = \int_{a_2}^{b_2} \left(\int_{\alpha(y)}^{\beta(y)} f(x,y)\,dx \right) dy$$

$\alpha, \beta \subseteq R \times R$
spojité na $\langle a_2, b_2 \rangle$

f ... ako v (1)

$$\iint_A (4-y)\,dx\,dy = \qquad A:\ -2 \le x \le 1,\quad 0 \le y \le 1-x$$



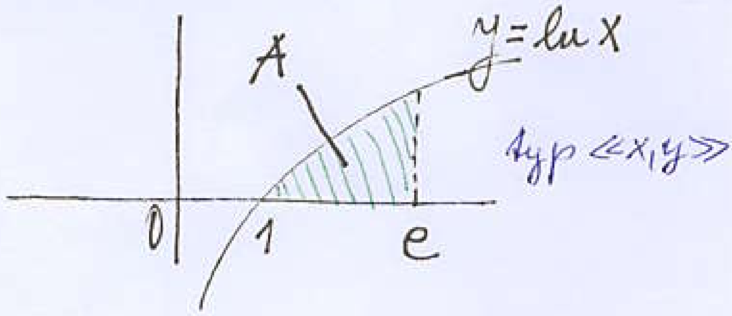
$$= \int_{-2}^{1} \left(\int_0^{1-x} (4-y)\,dy \right) dx =$$

$$= \int_{-2}^{1} \left[4y - \frac{y^2}{2} \right]_0^{1-x} dx =$$

$$= \int_{-2}^{1} \left(4(1-x) - \frac{1}{2}(1-x)^2 \right) dx = \int_{-2}^{1} \left(4 - 4x - \frac{1}{2}\left(1 - 2x + x^2\right) \right) dx =$$

$$= \int_{-2}^{1} \left(\frac{7}{2} - 3x - \frac{1}{2}x^2 \right) dx = \left[\frac{7}{2}x - 3\frac{x^2}{2} - \frac{1}{2}\frac{x^3}{3} \right]_{-2}^{1} = \frac{27}{2}$$

$$\iint_A x e^y\,dx\,dy \qquad A: \begin{cases} 1 \le x \le e \\ 0 \le y \le \ln x \end{cases}$$



$$= \int_1^e \left(\int_0^{\ln x} x e^y\,dy \right) dx =$$

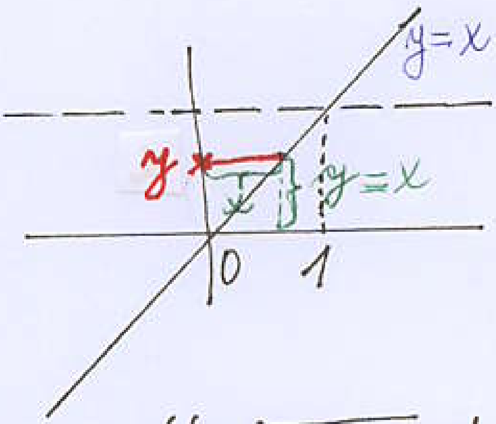
$$= \int_1^e x \left[e^y \right]_0^{\ln x} dx =$$

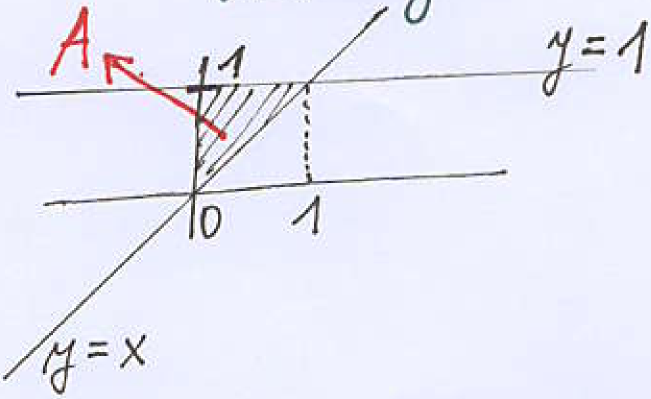
$$= \int_1^e x \left(e^{\ln x} - e^0 \right) dx = \int_1^e x(x-1)\,dx = \int_1^e (x^2 - x)\,dx =$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_1^e = \frac{e^3}{3} - \frac{e^2}{2} - \frac{1}{3} + \frac{1}{2}$$

$$\iint_A x\sqrt{y^2-x^2}\,dx\,dy \qquad A:\ \begin{matrix} 0\le y\le 1 \\ 0\le x\le y \end{matrix}$$

typ ≪y,x≫





$$\iint_A x\sqrt{y^2-x^2}\,dx\,dy = \int_0^1\left(\int_0^y x\sqrt{y^2-x^2}\,dx\right)dy =$$

$$= \int_0^1\left(\int_y^0 t(-t)\,dt\right)dy =$$

subst: $y^2-x^2=t^2$

$-2x\,dx = 2t\,dt$

$x\,dx = -t\,dt$

ak $x=0 \Rightarrow t=y$, $x=y \Rightarrow t=0$ } nové hranice pre t

$$= \int_0^1 (-1)\left[\frac{t^3}{3}\right]_y^0 dy =$$

$$= \int_0^1 (-1)\left(-\frac{y^3}{3}\right)dy = \frac{1}{3}\left[\frac{y^4}{4}\right]_0^1 = \frac{1}{12}$$

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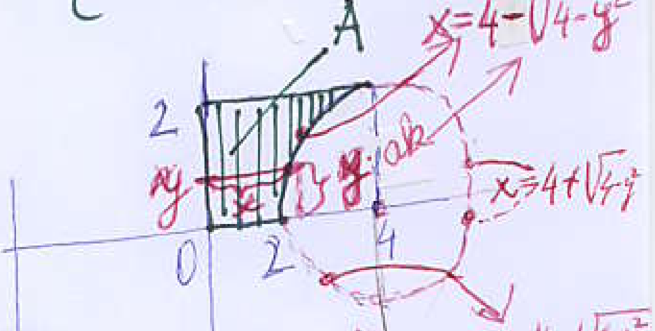
typ ≪y,x≫

$$\iint_A (x-4)y\,dx\,dy = \qquad A: \begin{cases} 0\le y\le 2 \\ 0\le x\le 4-\sqrt{4-y^2} \end{cases}$$

$$= \int_0^2\left(\int_0^{4-\sqrt{4-y^2}} (x-4)y\,dx\right)dy =$$

$$= \int_0^2 y\left[\frac{(x-4)^2}{2}\right]_0^{4-\sqrt{4-y^2}} dy =$$

$$= \int_0^2 y\left(\frac{4-y^2}{2}-8\right)dy = \int_0^2\left(-6y-\frac{y^3}{2}\right)dy = -14$$



$(x-4)^2+y^2=4$

$(x-4)^2 = 4-y^2$

$x-4 = \pm\sqrt{4-y^2}$

Výpočet trojných integrálov cez elementárne oblasti

(1) A je elementára oblasť typu $\ll x, y, z \gg$

$$A: \left.\begin{array}{l} \left.\begin{array}{r} a_1 \le x \le b_1 \\ \varphi(x) \le y \le \psi(x) \end{array}\right\} D \\ \alpha(x,y) \le z \le \beta(x,y) \end{array}\right\}$$

$\varphi, \psi \subseteq R \times R$ sú spojité na $\langle a_1, b_1 \rangle$

$\alpha, \beta \subseteq R^2 \times R$ sú spojité na D

$f \subseteq R^3 \times R$ je ohraničená a spojitá na A najviac okrem hranice A a konečne veľa vnútorných bodov množ. A.

Potom

$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iint_D \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz \right) dx\,dy =$$

$$= \int_{a_1}^{b_1} \left(\int_{\varphi(x)}^{\psi(x)} \left(\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz \right) dy \right) dx$$

(2) Elementárna oblasť A typu $\ll x, z, y \gg$

$$\left.\begin{array}{l} \left.\begin{array}{r} a_1 \le x \le b_1 \\ \varphi(x) \le z \le \psi(x) \end{array}\right\} D \\ \alpha(x,z) \le y \le \beta(x,z) \end{array}\right\}$$

$\varphi, \psi \subseteq R \times R$ spojité na $\langle a_1, b_1 \rangle$

$\alpha, \beta \subseteq R^2 \times R$ spojité na D

typu $\ll y, z, x \gg$

$$\left.\begin{array}{l} \left.\begin{array}{r} a_2 \le y \le b_2 \\ \varphi(y) \le z \le \psi(y) \end{array}\right\} D \\ \alpha(y,z) \le x \le \beta(y,z) \end{array}\right\}$$

$\varphi, \psi \subseteq R \times R$ spojité na $\langle a_2, b_2 \rangle$

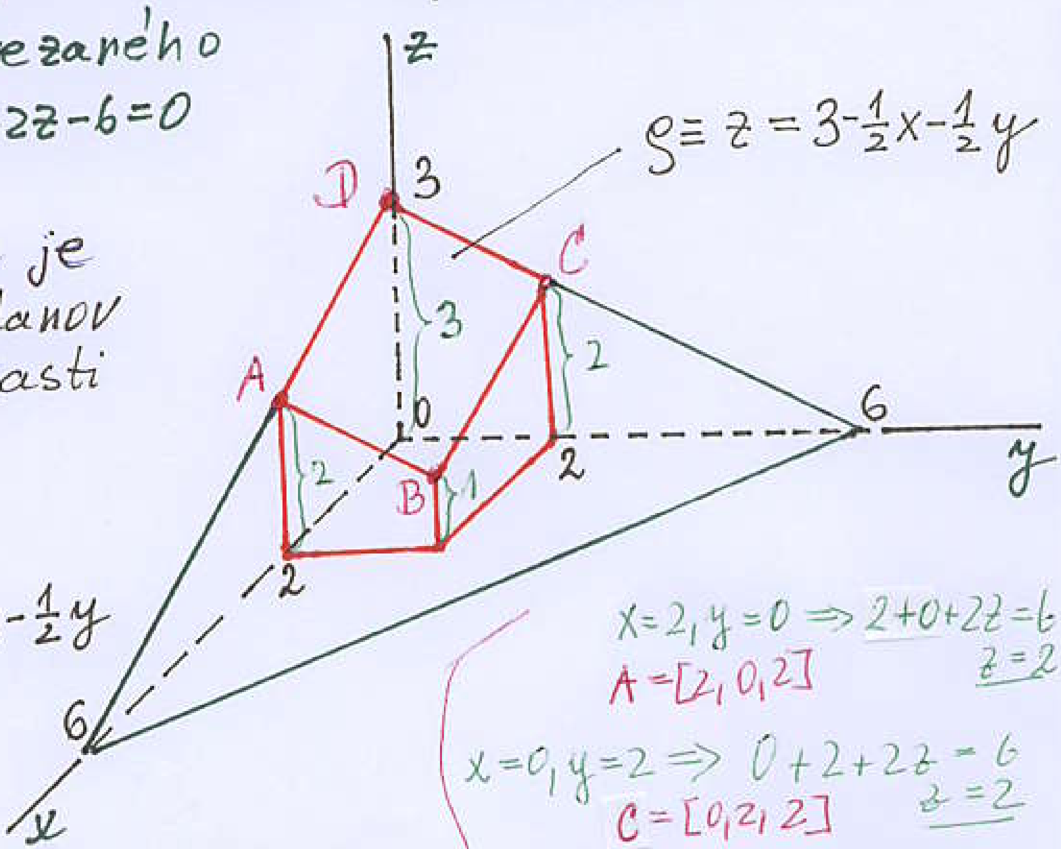
$\alpha, \beta \subseteq R^2 \times R$ spojité na D

ostatné typy podobne!

Vypočítajte objem kvádra s podstavou $\langle 0,2\rangle \times \langle 0,2\rangle$ v rovine x,y zrezaného rovinou $\varrho \equiv x+y+2z-6=0$

Hľadaný objem je trojrozmerný Jordanov obsah $C(A)$ oblasti

$$A: \begin{cases} 0 \le x \le 2 \\ 0 \le y \le 2 \\ 0 \le z \le 3-\frac{1}{2}x-\frac{1}{2}y \end{cases}$$



body v rovine $\varrho \equiv x+y+2z-6=0$

$x=2,\ y=0 \Rightarrow 2+0+2z=6$, $z=2$, $A=[2,0,2]$

$x=0,\ y=2 \Rightarrow 0+2+2z=6$, $z=2$, $C=[0,2,2]$

$x=0,\ y=0 \Rightarrow 0+0+2z=6$, $z=3$, $D=[0,0,3]$

$x=2,\ y=2 \Rightarrow 2+2+2z=6$, $z=1$, $B=[2,2,1]$

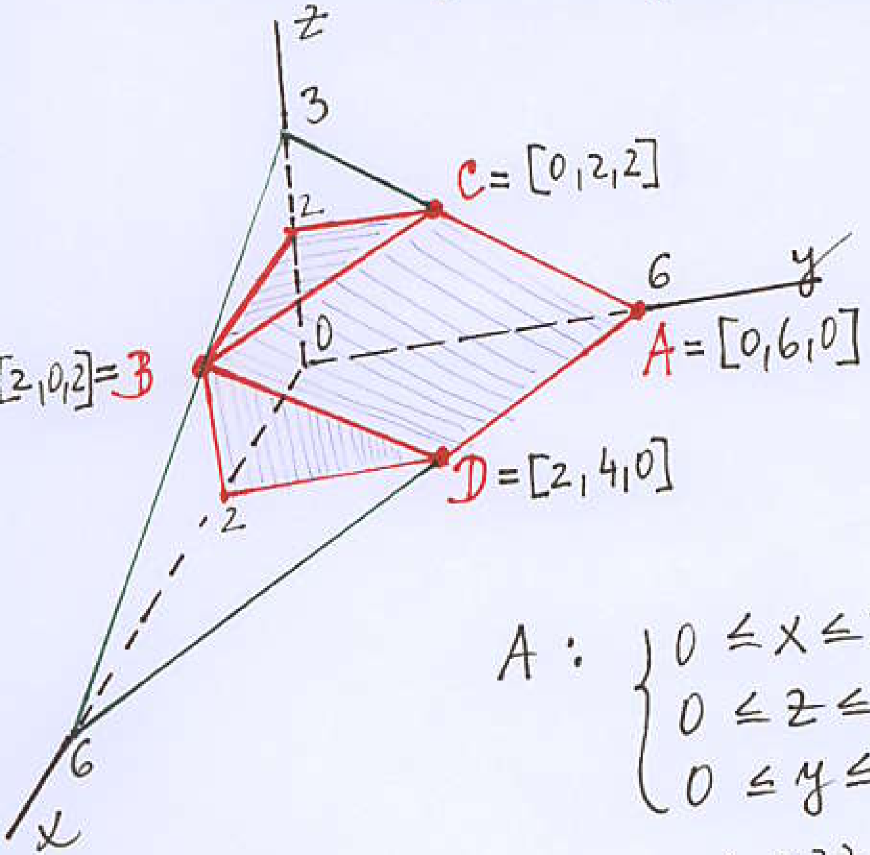
$$C(A)=\int_0^2\left(\int_0^2\left(\int_0^{3-\frac{1}{2}x-\frac{1}{2}y} dz\right)dy\right)dx=$$

$$=\int_0^2\left(\int_0^2\left(3-\frac{1}{2}x-\frac{1}{2}y\right)dy\right)dx=$$

$$=\int_0^2\left[3y-\frac{1}{2}xy-\frac{1}{4}y^2\right]_0^2 dx=\int_0^2(6-x-1)\,dx=$$

$$=\left[5x-\frac{x^2}{2}\right]_0^2=10-2=\boxed{8}$$

• Vypočítajte objem kvádra s podstavou $\langle 0,2\rangle \times \langle 0,2\rangle$ v rovine x, z a zrezaného rovinou $\varrho \equiv x+y+2z-6=0$



body v rovine ϱ:

$x=0, z=0 \Rightarrow y=6 \ldots A$

$x=2, z=0 \Rightarrow y=4 \ldots D$

$x=2, z=2 \Rightarrow y=0 \ldots B$

$x=0, z=2 \Rightarrow y=2 \ldots C$

$$A: \begin{cases} 0 \le x \le 2 \\ 0 \le z \le 2 \\ 0 \le y \le 6-x-2z \end{cases}$$

typ $\ll x, z, y \gg$

$$C(A) = \int_0^2 \left(\int_0^2 \left(\int_0^{6-x-2z} dy \right) dz \right) dx =$$

$$= \int_0^2 \left(\int_0^2 (6-x-2z)\, dz \right) dx =$$

$$= \int_0^2 \left[(6-x)z - 2\frac{z^2}{2} \right]_0^2 dx =$$

$$= \int_0^2 \left((6-x)2 - 4 \right) dx = \int_0^2 (8-2x)\, dx =$$

$$= \left[8x - 2\frac{x^2}{2} \right]_0^2 = 16 - 4 = 12$$